Assignment #9

Due on Wednesday, April 3, 2013

Read Section 5.3, on *Solving the One–dimensional Heat Equation,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`.

Background and Definitions

The **Error function**, Erf: $\mathbf{R} \to \mathbf{R}$, is defined by

$$\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\, ds, \qquad \text{for } x \in \mathbf{R}. \tag{1}$$

Do the following problems

1. Use the fact that

$$\int_0^\infty e^{-s^2}\, ds = \frac{\sqrt{\pi}}{2}$$

   to deduce that

   (a) $\lim\limits_{x\to\infty} \text{Erf}(x) = 1$; and

   (b) $\lim\limits_{x\to-\infty} \text{Erf}(x) = -1$.

2. Use the heat kernel to give a solution to the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}, & x \in \mathbf{R},\ t > 0; \\ u(x,0) = f(x), & x \in \mathbf{R}, \end{cases} \tag{2}$$

   where

$$f(x) = \begin{cases} 1, & \text{if } x \leqslant 0; \\ 0, & \text{if } x > 0. \end{cases} \tag{3}$$

   Express $u(x,t)$ in terms of the Error function in (1).

3. Use a mathematical software package to sketch the graph of $x \mapsto u(x,t)$ for several values of $t > 0$, where $u(x,t)$ is the solution to the initial value problem (2) with initial condition in (3) obtained in Problem 2.

4. Let $u(x,t)$ be the solution to the initial value problem (2) with initial condition in (3)obtained in Problem 2. Compute the following

   (a) $\lim\limits_{t\to 0^+} u(x,t)$, for $x = 0$ and for $x \neq 0$.

   (b) $\lim\limits_{x\to 0} u(x,t)$, for all $t > 0$.

5. Let $u(x,t)$ be the solution to the initial value problem (2) with initial condition in (3)obtained in Problem 2. Compute the following

   (a) $\lim\limits_{t\to\infty} u(x,t)$, for $x = 0$ and for $x \neq 0$.

   (b) $\lim\limits_{x\to\infty} u(x,t)$, for all $t > 0$.

   (c) $\lim\limits_{x\to-\infty} u(x,t)$, for all $t > 0$.