Exam 1

Friday, March 1, 2013

Name: ______________________________

This is a closed book exam. Show all significant work and justify all your answers. Use your own paper and/or the paper provided by the instructor. You have 50 minutes to work on the following 3 questions. Relax.

1. Assume the amount, $Q(t)$, of a substance in a compartment is given by a differentiable function of time, t. Assume also that the substance enters the compartment at a constant rate, $r > 0$, and leaves the compartment at a rate which is proportional to the amount present in the compartment, with constant of proportionality $\gamma > 0$.

   (a) State a conservation principle for the amount of substance in the compartment.

   (b) Use the conservation principle stated in the previous part to derive a differential equation model for the evolution in time of the amount of substance in the compartment.

   (c) Solve the differential equation model on the previous part and state what the model predicts about the amount of substance in the compartment in the long run.

2. The differential equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - EN, \tag{1}$$

   models a bacterial population that is being harvested at a rate proportional to the number of bacteria, N, in the culture. The parameter E is called the harvesting effort.

   (a) Give an interpretation to the model. In particular, what happens to the population in the absence of harvesting? What are the units for each of the parameters r, K and E?

(b) Nondimensionalize the differential equation in (1) by introducing dimensionless variables $u = \dfrac{N}{\mu}$ and $\tau = \dfrac{t}{\lambda}$ to obtain the dimensionless equation

$$\frac{du}{d\tau} = u\,(1-u) - \alpha u, \tag{2}$$

where α is a dimensionless parameters.

Express α in terms of the original parameters, and verify that it is dimensionless.

(c) Compute the equilibrium solutions of the equation in (2). Give interpretations for each of the equilibrium points and determine conditions under which the model in (2) yields biologically feasible equilibrium solutions. Express those conditions in terms of the original parameters. Determine the nature of the stability of the biological feasible equilibrium solutions.

3. The initial value problem for the partial differential equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + g'(u)\dfrac{\partial u}{\partial x} &= 0; \\ \\ u(x,0) &= f(x), \end{cases} \tag{3}$$

where $g(u) = u(1-u)$, was formulated in class as a model for traffic flow on a one–lane freeway.

(a) Give the equation for the characteristic curves of the partial differential equations in (3) and the differential equation that u satisfy when evaluated on a characteristic curve.

(b) Give an expression that defines a solution, $u(x,t)$, to (3) implicitly.

(c) For the initial density

$$f(x) = \begin{cases} 1, & \text{if } x < -1; \\ \dfrac{1}{2}(1-x), & \text{if } -1 \leqslant x < 1; \\ 0, & \text{if } x \geqslant 1, \end{cases} \tag{4}$$

sketch the characteristic curves of the partial differential equation in (3).

(d) Explain how the initial value problem (3) can be solved for the initial condition given in (4), and give a formula for $u(x,t)$.