Solutions to Exam #1

1. Assume the amount, $Q(t)$, of a substance in a compartment is given by a differentiable function of time, t. Assume also that the substance enters the compartment at a constant rate, $r > 0$, and leaves the compartment at a rate which is proportional to the amount present in the compartment, with constant of proportionality $\gamma > 0$.

   (a) State a conservation principle for the amount of substance in the compartment.

   ***Solution**:* Refer to the flow diagram in Figure 1.

   The conservation principle for Q in this case reads

   $$\frac{dQ}{dt} = \text{ Rate of } Q \text{ in } - \text{ Rate of } Q \text{ out}, \tag{1}$$

   where

   $$\text{Rate of } Q \text{ in } = r \tag{2}$$

   and

   $$\text{Rate of } Q \text{ out } = \gamma Q. \tag{3}$$

   □

   (b) Use the conservation principle stated in the previous part to derive a differential equation model for the evolution in time of the amount of substance in the compartment.

   ***Solution**:* Combine the equations in (1)–(3) to get the ordinary differential equation

   $$\frac{dQ}{dt} = r - \gamma Q. \tag{4}$$

   □

   (c) Solve the differential equation model on the previous part and state what the model predicts about the amount of substance in the compartment in the long run.

   ***Solution**:* Write the differential equation in (4) aa

   $$\frac{dQ}{dt} = -\gamma \left[Q - \frac{r}{\gamma} \right],$$

   and use separation of variables to get the general solution

   $$Q(t) = \frac{r}{\gamma} + ce^{-\gamma t}, \qquad \text{for } t \geqslant 0, \tag{5}$$

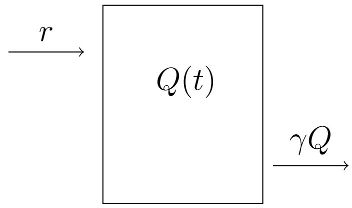

Figure 1: Flow Diagram for Problem 1

where c is a constant.

It follows from (5) and the assumption that $\gamma > 0$ that the amount the substance in the compartment tends towards the limiting value $\dfrac{r}{\gamma}$ in the long run. □

2. The differential equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - EN, \tag{6}$$

models a bacterial population that is being harvested at a rate proportional to the number of bacteria, N, in the culture. The parameter E is called the harvesting effort.

(a) Give an interpretation to the model. In particular, what happens to the population in the absence of harvesting? What are the units for each of the parameters r, K and E?

***Solution**:* The equation in (6) models a population that experience logistic growth in the absence of harvesting.

The intrinsic growth rate has units of 1 over time; the harvesting effort, E, also has units of 1 over time; the carrying capacity, K, has unit of number of cells. □

(b) Nondimensionalize the differential equation in (6) by introducing dimensionless variables $u = \dfrac{N}{\mu}$ and $\tau = \dfrac{t}{\lambda}$ to obtain the dimensionless equation

$$\frac{du}{d\tau} = u\,(1-u) - \alpha u, \tag{7}$$

where α is a dimensionless parameters.

Express α in terms of the original parameters, and verify that it is dimensionless.

***Solution**:* Apply the Chain Rule to obtain

$$\frac{du}{d\tau} = \frac{du}{dt} \cdot \frac{dt}{d\tau}, \tag{8}$$

where

$$\frac{du}{dt} = \frac{1}{\mu}\frac{dN}{dt} \tag{9}$$

and

$$\frac{dt}{d\tau} = \lambda. \tag{10}$$

Substituting (9) and (10) into (8) then yields

$$\frac{du}{d\tau} = \frac{\lambda}{\mu}\frac{dN}{dt}. \tag{11}$$

It follows from (11) and the differential equation in (6) that

$$\frac{du}{d\tau} = \frac{\lambda}{\mu} rN\left(1 - \frac{N}{K}\right) - \frac{\lambda E}{\mu}N,$$

which can be re–written as

$$\frac{du}{d\tau} = \lambda r\frac{N}{\mu}\left(1 - \frac{N/\mu}{K/\mu}\right) - \lambda E\frac{N}{\mu},$$

or

$$\frac{du}{d\tau} = \lambda ru\left(1 - \frac{u}{K/\mu}\right) - \lambda Eu. \tag{12}$$

Next, set

$$\lambda r = 1, \tag{13}$$

$$\frac{K}{\mu} = 1, \tag{14}$$

and

$$\alpha = \lambda E, \tag{15}$$

so that

$$\lambda = \frac{1}{r},$$

$$\mu = K,$$

and

$$\alpha = \frac{E}{r}. \tag{16}$$

Substituting (13)–(15) into (12) yields (7), where, according to (16), α is dimensionless because r and E have the same units. □

(c) Compute the equilibrium solutions of the equation in (7). Give interpretations for each of the equilibrium points and determine conditions under which the model in (7) yields biologically feasible equilibrium solutions. Express those conditions in terms of the original parameters. Determine the nature of the stability of the biological feasible equilibrium solutions.

***Solution**:* Set

$$f(u) = u\,(1-u) - \alpha u, \quad \text{for all } u \in \mathbb{R}; \tag{17}$$

then, equilibrium points of the differential equation in (7) are solutions to the equation

$$f(u) = 0,$$

or

$$\overline{u}_1 = 0 \quad \text{and} \quad \overline{u}_2 = 1 - \alpha. \tag{18}$$

In order to obtain a biologically feasible solution, we must require that

$$\alpha < 1,$$

or, in view of (16),

$$E < r;$$

in other words, the harvesting effort has to be less than the intrinsic growth rate of the population.

In order to determine the stability of the equilibrium solutions in (18) for the case $\alpha < 1$, we apply the Principle of Linearized Stability to the function, f, defined in (17), whose derivative is given by

$$f'(u) = 1 - \alpha - 2u, \quad \text{for all } u \in \mathbb{R}. \tag{19}$$

From $f'(\overline{u}_1) = 1 - \alpha > 0$, we conclude that $\overline{u}_1 = 0$ is unstable. From $f'(\overline{u}_2) = 1(1-\alpha) < 0$ we conclude that $\overline{u}_2 = 1 - \alpha$ is asymptotically stable. □

3. The initial value problem for the partial differential equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + g'(u)\dfrac{\partial u}{\partial x} & = & 0; \\ \\ u(x,0) & = & f(x), \end{cases} \tag{20}$$

where

$$g(u) = u(1-u), \tag{21}$$

was formulated in class as a model for traffic flow on a one–lane freeway.

(a) Give the equation for the characteristic curves of the partial differential equations in (20) and the differential equation that u satisfy when evaluated on a characteristic curve.

***Solution**:* The equation for the characteristic curves is given by

$$\frac{dx}{dt} = g'(u). \tag{22}$$

On characteristic curves, a solution, u, to the partial differential equation in (20) satisfies the ordinary differential equation

$$\frac{du}{dt} = 0,$$

which shows that u is constant along characteristic curves. We write

$$u(x,t) = \varphi(k), \tag{23}$$

where $\varphi(k)$ is the constant value of u on the characteristic indexed by k. Using the value for u in (23), the equation for the characteristic curves in (22) can be re–written as

$$\frac{dx}{dt} = g'(\varphi(k)). \tag{24}$$

Solving the differential equation in (24) yields the equation for the characteristic curves

$$x = g'(\varphi(k))t + k, \tag{25}$$

where the parameter k corresponds to the value on the x–axis on which the characteristic curves meet the x–axis.

□

(b) Give an expression that defines a solution, $u(x,t)$, to (20) implicitly.

***Solution**:* Solve for k in (25) and substitute into (23) to obtain the expression

$$u(x,t) = \varphi(x - g'(u(x,t))t), \tag{26}$$

which gives a solution of the partial differential equation in (20) implicitly. □

(c) For the initial density

$$f(x) = \begin{cases} 1, & \text{if } x < -1; \\ \dfrac{1}{2}(1-x), & \text{if } -1 \leqslant x < 1; \\ 0, & \text{if } x \geqslant 1, \end{cases} \tag{27}$$

sketch the characteristic curves of the partial differential equation in (20).

***Solution**:* Using the initial condition in (20) with f as given in (27), we obtain from (26) that

$$\varphi(x) = f(x), \quad \text{for all } x \in \mathbb{R},$$

so that (26) can now be re–written as

$$u(x,t) = f(x - g'(u(x,t))t). \tag{28}$$

Accordingly, the equation for the characteristic curves in (25) can now be re–written as

$$x = g'(f(k))t + k, \tag{29}$$

so that the characteristic curves will be straight lines in the xt–plane of slope $1/g'(f(k))$ going through $(k,0)$ for $k \in \mathbb{R}$, where $g'(u)$ is obtained from (21) as

$$g'(u) = 1 - 2u. \tag{30}$$

For instance, using (30), (27) and (29) we get that the equations for the characteristic curves for $k \leqslant -1$ are given by

$$x = -t + k, \quad \text{for } k \leqslant -1. \tag{31}$$

The curves described by (31) are straight lines with slope -1 going through $(k,0)$, for $k \leqslant -1$. Some of these are pictured in Figure 2. Similarly, for $k \geqslant 1$, the curves in (29) have equations

$$x = t + k, \quad \text{for } k \geqslant 1,$$

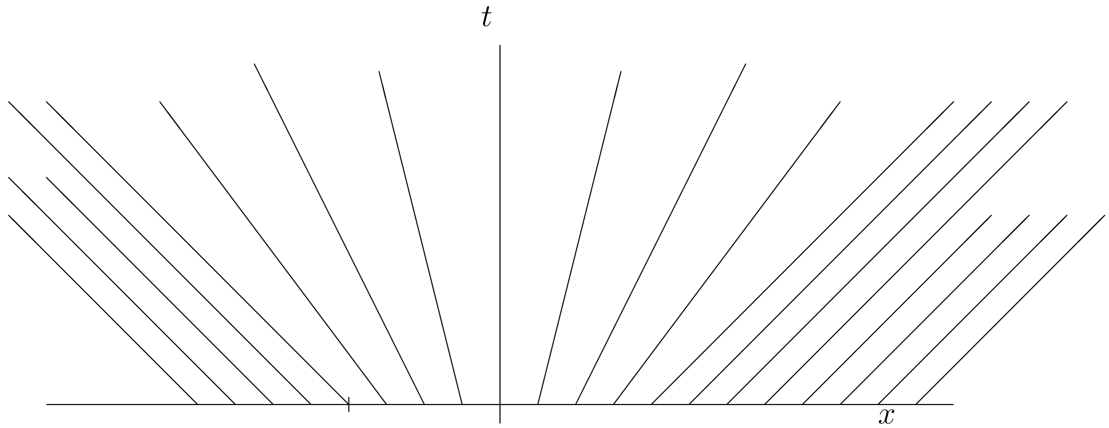

Figure 2: Characteristic Curves for Problem (20)

which are straight lines of slope 1 going through $(k, 0)$, for $k \geqslant 1$; some of these lines are also sketched in Figure 2.

For values of k between -1 and 1, the slopes of the lines in (29) are given by $1/g'(f(k))$, where $f(k)$ ranges from 1 at $k = -1$, to 0 at $k = 1$; so, according to (30), the slopes of the lines are negative and increase in absolute value to ∞ as k approaches 0. At $k = 0$, $f(k) = 1/2$, so that $g'(f(k)) = 0$, by virtue of (30), so that the characteristic curve will be $x = 0$, according to (29), or the t–axis. As k ranges from 0 to 1, the characteristic curves fan out from the t–axis to the line $x = t + 1$. A few of these curves are shown in Figure 2. □

(d) Explain how the initial value problem (20) can be solved for the initial condition given in (27), and give a formula for $u(x, t)$.

***Solution**:* Since the characteristic curves do not intersect for $t > 0$, the initial value problem in (20) with initial density given in (27) can always be solved by traveling back along the characteristic curves until it hits the x–axis at a point $(k, 0)$, and then reading the value of the initial density, $u(k, 0) = f(k)$, at that point. For example, if the point (x, t) lies in the region $x < -t - 1$, we see from Figure 2 that the characteristic curve containing the point (x, t) will meet the x–axis at some point $(k, 0)$ with $k < -1$; since, $f(k) = 1$ for $k < -1$, it follows from (28) that

$$u(x, t) = 1, \quad \text{for } x < -t - 1, \text{ and } t \geqslant 0. \tag{32}$$

Similarly, if $x \geqslant x + t$, then the characteristic curve containing (x, t) will meet the x–axis at some point $(k, 0)$ with $k \geqslant 1$; since $f(k) = 0$ for $k \geqslant 1$, it follows from (28) that

$$u(x,t) = 0, \qquad \text{for } x \geqslant t+1, \text{ and } t \geqslant 0. \tag{33}$$

For (x,t) lying in the region between the lines $x = -t-1$ and $x = t+1$, the characteristic curve containing the point will meet the x–axis at a point $(k,0)$ with $-1 \leqslant k \leqslant 1$. Since $f(k) = \dfrac{1}{2}(1-k)$ for those values of k, by (27), it follows from (28) that

$$u(x,t) = \frac{1}{2}[1 - (x - g'(u(x,t))t)], \qquad \text{for } -t-1 \leqslant x \leqslant t+1. \tag{34}$$

Using (30), we can re–write (34) as

$$u(x,t) = \frac{1-x+t}{2} - u(x,t)t, \qquad \text{for } -t-1 \leqslant x \leqslant t+1. \tag{35}$$

Solving for $u(x,t)$ in (35) yields

$$u(x,t) = \frac{1-x+t}{2(1+t)}, \qquad \text{for } -t-1 \leqslant x \leqslant t+1. \tag{36}$$

Finally, putting together the results in (32), (33) and (36), we obtain the following formula for $u(x,t)$:

$$u(x,t) = \begin{cases} 1, & \text{for } x < -t-1; \\ \dfrac{1-x+t}{2(1+t)}, & \text{for } -t-1 \leqslant x \leqslant t+1; \\ 0, & \text{for } x > t+1, \end{cases}$$

for $t \geqslant 0$. $\square$