Assignment #13

Due on Monday, March 31, 2014

Read Section 4.2 on *Using Symmetry to Solve PDEs* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read Section A.2, on *Self-Similar Solutions*, in the text, pp. 287–294.

Do the following problems

1. Assume that u solves Laplace's equation in $\mathbb{R}^2$. For fixed $(\overline{x}, \overline{y})$ in $\mathbb{R}^2$, define

$$\left\{ \begin{array}{rcl} \xi & = & x - \overline{x}; \\ \eta & = & y - \overline{y}, \end{array} \right. \tag{1}$$

and set

$$v(\xi, \eta) = u(x, y),$$

where x and y are given in terms of ξ and η by inverting the transformation equations in (1). Apply the Chain Rule to verify that

$$v_{\xi\xi} + v_{\eta\eta} = 0.$$

We say that Laplace's equation is **translation invariant**.

2. In class and in the lecture notes we showed that dilation–invariant solutions of the one–dimensional heat equation,

$$u_t = k u_{xx}, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0,$$

where k is the thermal diffusivity, are of the form

$$u(x, t) = c_1 \int_0^{x/\sqrt{t}} e^{-z^2/4k}\, dz + c_2, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{2}$$

and constants c_1 and c_2.

Use the formula in (2) to find a solution to the initial–boundary-value problem

$$\begin{cases} u_t - k u_{xx} = 0, & \text{for } x > 0, t > 0, \\ u(x, 0) = 0, & \text{for all } x > 0; \\ u(0, t) = 1, & \text{for all } t > 0. \end{cases} \tag{3}$$

Note: The initial condition in (3) is to be understood as

$$\lim_{t \to 0^+} u(x, t) = 0, \qquad \text{for all } x > 0.$$

3. Verify that the function

$$p(x,t) = \frac{e^{-x^2/4Dt}}{\sqrt{4\pi D t}}, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{4}$$

solves the one–dimensional diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0.$$

4. Let p be as defined in (4). Show that

$$\int_{-\infty}^{\infty} p(x-y,t)\ dy = 1, \qquad \text{for all } x \in \mathbb{R} \text{ and all } t > 0.$$

5. Let p be as defined in (4). Show the following:

   (a) If $x \neq 0$, then $\lim\limits_{t\to 0^+} p(x,t) = 0$.

   (b) If $x = 0$, then $\lim\limits_{t\to 0^+} p(x,t) = +\infty$.