Assignment #15

Due on Friday, April 11, 2014

Read Section 5.2 on *Solving the Dirichlet Problem in the Unit Disk* in the class lecture notes at`http://pages.pomona.edu/~ajr04747/`

Read Section 2.1, on *Separations of Variable*, in the text, pp. 141–167.

Read Section 2.5, on *Fourier Series and Green's Functions*, in the text, pp. 182–194.

Do the following problems

1. Let R denote the open square $\{(x, y) \in \mathbb{R}^2 \mid 0 < x < \pi, 0 < y < \pi\}$. Find all values of λ for which the following BVP

$$\begin{cases} -(u_{xx} + u_{yy}) &= \lambda u \quad \text{in } R; \\ \qquad\qquad u &= 0, \quad \text{on } \partial R, \end{cases} \tag{1}$$

   has nontrivial solutions. Those values are called **eigenvalues** of problem (1).

   For each eigenvalue give the corresponding nontrivial solutions; these are called **eigenfuctions**.

2. Let Ω denote a path–connected, bounded open subset of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$.

   Suppose that the BVP

$$\begin{cases} -(u_{xx} + u_{yy}) &= \lambda u \quad \text{in } \Omega; \\ \qquad\qquad u &= 0, \quad \text{on } \partial \Omega, \end{cases}$$

   has a nontrivial solution. Show that λ must be positive.

3. **Green's Second Identity**. Use the result of Problem 4 in Assignment #6 to derive **Green's Second Identity**:

   Let Ω denote an open region in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ and let u and v denote scalar functions in $C^2(\Omega) \cap C(\overline{\Omega})$. Then,

$$\iiint_\Omega [u\Delta v - v\Delta u] \; dV = \iint_{\partial\Omega} \left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right] \; dA,$$

where $\dfrac{\partial u}{\partial n}$ denotes the directional derivative of u in the direction of the outward unit normal on the boundary of Ω.

State the two–dimensional analogue of this result.

4. **Eigenvalues and Eigenfunctions of the Laplacian**. Let Ω denote an open, path–connected, bounded subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. If the BVP

$$\begin{cases} -\Delta u &= \lambda u \quad \text{in } \Omega; \\ u &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

has nontrivial solutions, we call λ and **eigenvalue** of (2); a corresponding non-trivial solution is called an **eigenfunction** for that value of λ.

Use Green's Second Identity (derived in Problem 3) to shown that if u and v are eigenfunctions corresponding to distinct eigenvalues of (2), then

$$\int_\Omega uv = 0.$$

5. **Green's Function for the Half Plane**. Define $G\colon \mathbb{R} \times (0, \infty) \to \mathbb{R}$ by

$$G(x, y) = \frac{1}{\pi} \frac{y}{x^2 + y^2}, \qquad \text{for } x \in \mathbb{R} \text{ and } y > 0.$$

Verify the following properties of G:

(a) $\Delta G = 0$ in $\mathbb{R} \times (0, \infty) = \{(x, y) \in \mathbb{R}^2 \mid -\infty < x < \infty, y > 0\}$, the upper half plane.

(b) $\displaystyle\int_{-\infty}^{\infty} G(x - s, y)\, ds = 1$ for all $x \in \mathbb{R}$ and all $y > 0$.

(c) $\displaystyle\lim_{y\to 0^+} G(x, y) = 0$ for $x \neq 0$ and $\displaystyle\lim_{y\to 0^+} G(x, y) = +\infty$ for $x = 0$.