Assignment #9

Due on Monday, February 24, 2014

Read Chapter 3 on *Classification of PDEs* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read pages 1–13 in the text.

Do the following problems

1. Define $N\colon C^1(\mathbb{R}^2) \to C(\mathbb{R}^2)$ by
$$N(u)(x,t) = \frac{\partial}{\partial t}[u(x,t)] + u(x,t)\frac{\partial}{\partial x}[u(x,t)], \quad \text{for all } (x,t) \in \mathbb{R}^2,$$
and all $u \in C^1(\mathbb{R}^2)$. Show that N is not a linear operator.

2. Let u and v be two solutions of the linear PDE
$$Lu = f \tag{1}$$
in a linear space of differentiable functions $\mathcal{V}$. Put $w = u - v$. Show that w solves the homogeneous PDE
$$Lw = 0. \tag{2}$$
Show that, if w is a solution of the homogeneous PDE in (2) and u is a solution of (1), then $u + w$ solves the PDE in (1).

3. Let R denote an open subset of $\mathbb{R}^3$ and $g \in C^2(R) \cap C(\overline{R})$.

   Suppose that $v \in C^2(R) \cap C(\overline{R})$ is a solution to the following Dirichlet problem for Laplace's equation:
$$\begin{cases} -\Delta v = 0, & \text{in } R; \\ v = g, & \text{on } \partial R. \end{cases}$$
   Show that the function $u = v - g$ is a solution to the following Dirichlet problem for Poisson's equation:
$$\begin{cases} -\Delta u = f, & \text{in } R; \\ u = 0, & \text{on } \partial R, \end{cases}$$
   where $f = \Delta g$.

4. In this problem and the next we consider the following Dirichlet problem for Poisson's equation:
$$\begin{cases} -\Delta u = f, & \text{in } R; \\ u = 0, & \text{on } \partial R, \end{cases} \tag{3}$$
where R is a bounded open subset of $\mathbb{R}^3$ with smooth boundary, ∂R, and f is a known function that is continuous on $\overline{R}$, the closure of R.

   Denote by $C_o^2(R)$ the space of functions
$$\{u \in C^2(R) \cap C(\overline{R}) \mid u = 0 \text{ on } \partial R\};$$
that is, $C_o^2(R)$ is the space of C^2 functions in R that vanish on the boundary of R.

   Define the functional $J\colon C_o^2(R) \to \mathbb{R}$ by
$$J(u) = \frac{1}{2}\iiint_R |\nabla u|^2 \, dV - \iiint_R fu \, dV, \quad \text{for all } u \in C_o^2(R). \tag{4}$$
For given $u \in C_o^2(R)$ and $\varphi \in C_c^\infty(R)$, define $h\colon \mathbb{R} \to \mathbb{R}$ by
$$h(t) = J(u + t\varphi), \quad \text{for } t \in \mathbb{R}.$$
Compute $h'(t)$ for all t in R and show that
$$h'(0) = \iiint_R \nabla u \cdot \nabla \varphi \, dV - \iiint_R f\varphi \, dV.$$
Show that if u is a minimizer of the functional J defined in (4) in the space $C_o^2(R)$, then
$$\iiint_R \nabla u \cdot \nabla \varphi \, dV - \iiint_R f\varphi \, dV = 0, \quad \text{for all } \varphi \in C_c^\infty(R). \tag{5}$$

5. Show that, if (5) holds true for $u \in C_o^2(R)$, then u is a solution of the BVP in (3).