Exam 1

Due on Friday, March 14, 2014

Name: ________________________________

This is an open–notes, open text exam; you may consult your own notes, or the class notes in my courses website at `http://pages.pomona.edu/~ajr04747/`, or the text for the course.

Students are expected to work individually on these problems. You may not consult with anyone.

Show all significant work and provide reasoning for all your assertions.

Write your name on this page and staple it to your solutions. Turn in your solutions at the start of class on **Friday, March 14, 2014.**

I have read and agree to these instructions. Signature: ______________________

1. Consider the system of linear first order PDEs

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = 0; \\ \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = 0, \end{cases} \tag{1}$$

where u and v denote C^2 functions defined in an open region, R, of $\mathbb{R}^2$. The system of PDEs in (1) is known as the Cauchy–Riemann equations.

(a) Assume that $u, v \in C^2(R)$. Verify that u and v both solve Laplace's equation in R.

(b) Assume that $u, v \in C^2(R) \cap C(\overline{R})$ and that R is bounded with smooth boundary, ∂R. Show that there can be at most one solution to the system in (1) satisfying the boundary conditions

$$\begin{cases} u(x,y) = f(x,y), & \text{for } (x,y) \in \partial R; \\ \\ v(x,y) = g(x,y), & \text{for } (x,y) \in \partial R, \end{cases}$$

where f and g are given functions that are defined and continuous on a neighborhood of ∂R.

2. A subset R of $\mathbb{R}^2$ is said to be **path–connected** if, for any two points, (x_o, y_o) and (x_1, y_1), in R there exists a C^1 path $\gamma\colon [0,1] \to \mathbb{R}^2$ such that

$$\gamma(0) = (x_o, y_o), \ \gamma(1) = (x_1, y_1) \ \text{ and } \ \gamma(t) = (x(t), y(t)) \in R \text{ for all } t \in [0,1].$$

   (a) Assume that R is open and path–connected. Let $u \in C^1(R)$ be a solution of the system of first–order PDEs

$$\begin{cases} \dfrac{\partial u}{\partial x} &= 0, \quad \text{in } R; \\[2ex] \dfrac{\partial u}{\partial y} &= 0, \quad \text{in } R. \end{cases}$$

   Prove that u must be constant in R.

   (b) Assume that R is open and path–connected. Let $u \in C^1(R)$ satisfy

$$\iint_R |\nabla u|^2 \; dxdy = 0.$$

   Prove that u must be constant in R.

   (c) Assume that R is open and path–connected. Suppose that $u \in C_c^\infty(R)$ satisfies

$$\iint_R |\nabla u|^2 \; dxdy = 0.$$

   What can you conclude about u?

3. In this problem we study the following initial value problem for a quasilinear first–order PDE:

$$\begin{cases} u_t + uu_x &= 1, \qquad \text{for } x \in \mathbb{R}, t > 0; \\ u(x,0) &= f(x), \quad \text{for } x \in \mathbb{R}, \end{cases} \tag{2}$$

   where f is a given C^1 function.

   (a) Use the method of characteristic curves to find an implicit solution to the initial value problem (2).

   (b) Compute a solution to the IVP in (2) for the special case in which $f(x) = x$ for all $x \in \mathbb{R}$; that is, give a formula for computing $u(x,t)$ for $x \in \mathbb{R}$ and $t > 0$.

4. In Assignment #4 you derived the one dimensional heat equation

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0, \qquad \text{for } 0 < x < L, t > 0, \tag{3}$$

which models the flow of heat in a cylindrical rod of length L, constant–cross sectional area, and thermal diffusivity k. The value $u(x,t)$ gives the temperature in the cross–section of the rod at x and time t.

In this problem we study the initial–boundary–value problem for the PDE in (3):

$$\begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial^2 u}{\partial x^2} = 0, & \text{for } 0 < x < L, t > 0, \\ u(x,0) = f(x), & \text{for all } x \in [0,L]; \\ u(0,t) = U_o(t), & \text{for all } t; \\ u(L,t) = U_{L}(t), & \text{for all } t, \end{cases} \tag{4}$$

where f, U_o and U_L are given continuous functions of a single variable.

(a) For the case in which $U_o(t) = U_L(t) = 0$ for all t in (4), we obtain the initial–boundary–value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial^2 u}{\partial x^2} = 0, & \text{for } 0 < x < L, t > 0, \\ u(x,0) = f(x), & \text{for all } x \in [0,L]; \\ u(0,t) = 0, & \text{for all } t; \\ u(L,t) = 0, & \text{for all } t. \end{cases} \tag{5}$$

Define the total energy: $E(t) = \dfrac{1}{2}\displaystyle\int_0^L u^2\, dx$, for all t.

Assume that u solves the initial–boundary–value problem in (5). Show that $E(t) \leqslant E(0)$, for all $t \geqslant 0$, so that

$$\int_0^L [u(x,t)]^2\, dx \leqslant \int_0^L |f(x)|^2\, dx, \qquad \text{for all } t \geqslant 0.$$

Suggestion: Compute the rate of change of total energy, $E'(t)$, for all t.

(b) Show that, if $f(x) = 0$ for all $x \in [0,L]$ in (5), then any solution to the initial–boundary–value problem in (5) must be 0 for all x and all t.

(c) Prove that the initial–boundary–value problem in (4) can have at most one solution.

5. In this problem we consider the following nonlinear boundary value problem:

$$\begin{cases} -\Delta u = g(u), & \text{in } R; \\ u = 0, & \text{on } \partial R, \end{cases} \tag{6}$$

where R is a bounded open subset of $\mathbb{R}^3$ with smooth boundary, ∂R, and

$$g\colon \mathbb{R} \to \mathbb{R}$$

is a continuous real valued function of a single real variable. Define $G\colon \mathbb{R} \to \mathbb{R}$ by

$$G(\xi) = \int_0^{\xi} g(s)\, ds, \quad \text{for all } \xi \in \mathbb{R}.$$

Denote by $C_o^2(R)$ the space of functions $\{u \in C^2(R) \cap C(\overline{R}) \mid u = 0 \text{ on } \partial R\}$; that is, $C_o^2(R)$ is the space of C^2 functions in R that vanish on the boundary of R.

Define the functional $J\colon C_o^2(R) \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\iiint_R |\nabla u|^2\, dV - \iiint_R G(u)\, dV, \quad \text{for all } u \in C_o^2(R). \tag{7}$$

(a) For given $u \in C_o^2(R)$ and $\varphi \in C_c^\infty(R)$, define $h\colon \mathbb{R} \to \mathbb{R}$ by

$$h(t) = J(u + t\varphi), \quad \text{for } t \in \mathbb{R}.$$

Compute $h'(t)$ for all t in $\mathbb{R}$ and show that

$$h'(0) = \iiint_R \nabla u \cdot \nabla \varphi\, dV - \iiint_R g(u)\varphi\, dV.$$

(b) Show that if u is a minimizer of the functional J defined in (7) in the space $C_o^2(R)$, then

$$\iiint_R \nabla u \cdot \nabla \varphi\, dV - \iiint_R g(u)\varphi\, dV = 0, \quad \text{for all } \varphi \in C_c^\infty(R). \tag{8}$$

(c) Show that, if (8) holds true for $u \in C_o^2(R)$, then u is a solution of the BVP in (6).