Solutions to Exam #1

1. Consider the system of linear first order PDEs

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} & = & 0; \\ & & \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} & = & 0, \end{cases} \tag{1}$$

where u and v denote C^2 functions defined in an open region, R, of $\mathbb{R}^2$. The system of PDEs in (1) is known as the Cauchy–Riemann equations.

(a) Assume that $u, v \in C^2(R)$. Verify that u and v both solve Laplace's equation in R.

Solution: Differentiate the first equation in (1) with respect to x and the second one with respect to y to get

$$u_{xx} = v_{yx} \tag{2}$$

and

$$u_{yy} = -v_{xy}, \tag{3}$$

respectively. Then, adding the equations in (2) and (3), and using the fact mixed second partial derivatives of C^2 functions in $\mathbb{R}^2$ are equal,

$$u_{xx} + u_{yy} = 0, \tag{4}$$

which shows that u is harmonic in $\mathbb{R}^2$. Similar calculations show that v is also harmonic in $\mathbb{R}^2$. □

(b) Assume that $u, v \in C^2(R) \cap C(\overline{R})$ and that R is bounded with smooth boundary, ∂R. Show that there can be at most one solution to the system in (1) satisfying the boundary conditions

$$\begin{cases} u(x, y) & = & f(x, y), & \text{for } (x, y) \in \partial R; \\ & & & \\ v(x, y) & = & g(x, y), & \text{for } (x, y) \in \partial R, \end{cases} \tag{5}$$

where f and g are given functions that are defined and continuous on a neighborhood of ∂R.

***Solution**:* According to (4), u is a solution of the Dirichlet boundary value problem

$$\begin{cases} u_{xx} + u_{yy} = 0, & \text{in } R; \\ u = f, & \text{on } \partial R. \end{cases} \tag{6}$$

By the result of Problem 3 in Assignment #7, the BVP in (6) can have at most one solution. Similarly, the Dirichlet BVP

$$\begin{cases} v_{xx} + v_{yy} = 0, & \text{in } R; \\ v = g, & \text{on } \partial R, \end{cases}$$

can have at most one solution. Hence, there can be at most one solution to the system in (1) satisfying the boundary conditions in (5). □

2. A subset R of $\mathbb{R}^2$ is said to be **path–connected** if, for any two points, (x_o, y_o) and (x_1, y_1), in R there exists a C^1 path $\gamma\colon [0,1] \to \mathbb{R}^2$ such that

$$\gamma(0) = (x_o, y_o), \ \gamma(1) = (x_1, y_1) \ \text{ and } \ \gamma(t) = (x(t), y(t)) \in R \text{ for all } t \in [0,1].$$

(a) Assume that R is open and path–connected. Let $u \in C^1(R)$ be a solution of the system of first–order PDEs

$$\begin{cases} \dfrac{\partial u}{\partial x} = 0, & \text{in } R; \\[2ex] \dfrac{\partial u}{\partial y} = 0, & \text{in } R. \end{cases} \tag{7}$$

Prove that u must be constant in R.

***Solution**:* Assume that R is path connected and that $u \in C^1(R)$ solves the system in (7). Let (x_o, y_o) be a fixed point in R. Then, since R is path is connected, for any $(x, y) \in R$ there exists a C^1 path $\gamma\colon [0,1] \to \mathbb{R}^2$ such that

$$\gamma(t) = (x(t), y(t)) \in R \text{ for all } t \in [0,1]. \tag{8}$$

and

$$\gamma(0) = (x_o, y_o) \ \text{ and } \ \gamma(1) = (x, y). \tag{9}$$

Let $h(t) = u(\gamma(t)) = u(x(t), y(t))$ for all $t \in [0,1]$. By the Chain Rule, $h \in C^1(0,1)$ and

$$h'(t) = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}, \qquad \text{for all } t \in (0,1). \tag{10}$$

It follows from (8), (7) and (10) that $h'(t) = 0$ for all $t \in (0,1)$. Thus, h is constant on $(0,1)$ and, by continuity of h on $[0,1]$, $h(1) = h(0)$, which implies that

$$u(\gamma(1)) = u(\gamma(0)). \tag{11}$$

It follows from (11) and (9) that

$$u(x,y) = u(x_o, y_o). \tag{12}$$

Since (x,y) is an arbitrary point in R, it follows from (12) that u is constant in R. □

(b) Assume that R is open and path–connected. Let $u \in C^1(R)$ satisfy

$$\iint_R |\nabla u|^2 \, dxdy = 0. \tag{13}$$

Prove that u must be constant in R.

***Solution**:* Assume that R is open and path–connected and $u \in C^1(R)$. Then, the integrand in (13) is continuous and nonnegative. It then follows from (13) that

$$|\nabla u|^2 = 0 \text{ in } R,$$

or

$$u_x^2 + u_y^2 = 0 \text{ in } R,$$

which implies that

$$u_x = u_y = 0 \text{ in } R.$$

It then follows from the result in part (a) that u is constant in R. □

(c) Assume that R is open and path–connected. Suppose that $u \in C_c^\infty(R)$ satisfies

$$\iint_R |\nabla u|^2 \, dxdy = 0. \tag{14}$$

What can you conclude about u?

***Solution**:* Assume that that R is open and path–connected, $u \in C_c^\infty(R)$ and (14) holds true. It follows from (14) and the result of part (b) that u is constant in R. Now, since u has support in R, $u = 0$ on ∂R. Thus, by continuity of u, $u(x,y) = 0$ for all $(x,y) \in R$. □

3. In this problem we study the following initial value problem for a quasilinear first–order PDE:

$$\begin{cases} u_t + u u_x = 1, & \text{for } x \in \mathbb{R}, t > 0; \\ u(x,0) = f(x), & \text{for } x \in \mathbb{R}, \end{cases} \tag{15}$$

where f is a given C^1 function.

(a) Use the method of characteristic curves to find an implicit solution to the initial value problem (15).

***Solution**:* The equation for the characteristic curves of the partial differential equation in (15) is

$$\frac{dx}{dt} = u. \tag{16}$$

Along characteristic curves, u satisfies the ordinary differential equation

$$\frac{du}{dt} = 1,$$

which can be solved to yield

$$u(x,t) = t + F(\xi), \tag{17}$$

where $F(\xi)$ depends on the characteristic curve in (16) indexed by ξ. Substituting (17) into (16) yields the ODE

$$\frac{dx}{dt} = t + F(\xi),$$

which can be solved to yield

$$x = \frac{t^2}{2} + F(\xi)t + \xi. \tag{18}$$

Solving for ξ in (18) and substituting into (17) yields

$$u(x,t) = t + F\left(x - \frac{t^2}{2} - (u(x,t) - t))t\right),$$

where we have used (17), or

$$u(x,t) = t + F\left(x + \frac{t^2}{2} - tu(x,t)\right), \tag{19}$$

which gives $u(x,t)$ implicitly.

Using the initial condition in (refExam1Prob3Eqn05), we obtain from (19) that

$$F(x) = f(x), \quad \text{for all } x,$$

so that

$$u(x,t) = t + f\left(x + \frac{t^2}{2} - tu(x,t)\right), \quad \text{for } x \in \mathbb{R},\ t \geqslant 0. \tag{20}$$

□

(b) Compute a solution to the IVP in (15) for the special case in which $f(x) = x$ for all $x \in \mathbb{R}$; that is, give a formula for computing $u(x,t)$ for $x \in \mathbb{R}$ and $t > 0$.

***Solution**:* Using $f(x) = x$ for all x in (20) we obtain

$$u(x,t) = t + x + \frac{t^2}{2} - tu(x,t), \qquad \text{for } x \in \mathbb{R},\ t \geqslant 0. \tag{21}$$

Solving for $u(x,t)$ in (21) then yields

$$u(x,t) = \frac{2x + 2t + t^2}{2(1+t)}, \qquad \text{for } x \in \mathbb{R} \text{ and } t \geqslant 0.$$

□

4. In Assignment #4 you derived the one–dimensional heat equation

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0, \qquad \text{for } 0 < x < L, t > 0, \tag{22}$$

which models the flow of heat in a cylindrical rod of length L, constant–cross sectional area, and thermal diffusivity k. The value $u(x,t)$ gives the temperature in the cross–section of the rod at x and time t.

In this problem we study the initial–boundary–value problem for the PDE in (22):

$$\begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial^2 u}{\partial x^2} = 0, & \text{for } 0 < x < L, t > 0, \\ u(x,0) = f(x), & \text{for all } x \in [0,L]; \\ u(0,t) = U_o(t), & \text{for all } t; \\ u(L,t) = U_L(t), & \text{for all } t, \end{cases} \tag{23}$$

where f, U_o and U_L are given continuous functions of a single variable.

(a) For the case in which $U_o(t) = U_L(t) = 0$ for all t in (23), we obtain the initial–boundary–value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial^2 u}{\partial x^2} = 0, & \text{for } 0 < x < L, t > 0, \\ u(x,0) = f(x), & \text{for all } x \in [0,L]; \\ u(0,t) = 0, & \text{for all } t; \\ u(L,t) = 0, & \text{for all } t. \end{cases} \tag{24}$$

Define the total energy: $E(t) = \dfrac{1}{2}\displaystyle\int_0^L u^2\, dx,$ for all t.

Assume that u solves the initial–boundary–value problem in (24). Show that $E(t) \leqslant E(0)$, for all $t \geqslant 0$, so that

$$\int_0^L [u(x,t)]^2\, dx \leqslant \int_0^L |f(x)|^2\, dx, \qquad \text{for all } t \geqslant 0.$$

Suggestion: Compute the rate of change of total energy, $E'(t)$, for all t.

***Solution**:* Differentiating under the integral sign we obtain

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{2}\int_0^L \frac{\partial}{\partial t}[u^2]\, dx \\ &= \frac{1}{2}\int_0^L 2uu_t\, dx, \end{aligned}$$

so that

$$\frac{dE}{dt} = \int_0^L uu_t\, dx.$$

Thus, using the assumption that u solves $u_t = ku_{xx}$,

$$\frac{dE}{dt} = k\int_0^L uu_{xx}\, dx. \tag{25}$$

Next, integrate by parts on the right–hand side of (25) to get

$$\frac{dE}{dt} = kuu_x\Big|_0^L - k\int_0^L u_x u_x\, dx$$

or

$$\frac{dE}{dt} = -k\int_0^L u_x^2\, dx, \tag{26}$$

where we have used the boundary conditions in (24).

It follows from (26) and the assumption that $k > 0$ that $E'(t) \leqslant 0$ for all t; so that $E(t)$ is decreasing as t increases. Consequently,

$$E(t) \leqslant E(0), \qquad \text{for all } t \geqslant 0.$$

Thus, using the definition of $E(t)$,

$$\frac{1}{2}\int_0^L [u(x,t)]^2\, dx \leqslant \frac{1}{2}\int_0^L [u(x,0)]^2\, dx, \qquad \text{for all } t \geqslant 0.$$

or, by virtue of the initial condition in (24)

$$\int_0^L [u(x,t)]^2\ dx \leqslant \int_0^L [f(x)]^2\ dx, \qquad \text{for all } t \geqslant 0, \tag{27}$$

which was to be shown. □

(b) Show that, if $f(x) = 0$ for all $x \in [0, L]$ in (24), then any solution to the initial–boundary–value problem in (24) must be 0 for all x and all t.

Solution: Suppose that the initial condition f in (24) satisfies $f(x) = 0$ for all $x \in [0, L]$; it then follows from (27) that, for any solution of the problem (24) with that initial condition,

$$\int_0^L [u(x,t)]^2\ dx \leqslant 0, \qquad \text{for all } t \geqslant 0,$$

from which we get that

$$\int_0^L [u(x,t)]^2\ dx = 0, \qquad \text{for all } t \geqslant 0, \tag{28}$$

It follows from (28) and the continuity of u that any solution of the initial–boundary–value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial^2 u}{\partial x^2} = 0, & \text{for } 0 < x < L, t > 0, \\ u(x,0) = 0, & \text{for all } x \in [0, L]; \\ u(0,t) = 0, & \text{for all } t; \\ u(L,t) = 0, & \text{for all } t. \end{cases} \tag{29}$$

must be $u(x,t) = 0$ for $0 \leqslant x \leqslant L$ and $t \geqslant 0$. □

(c) Prove that the initial–boundary–value problem in (23) can have at most one solution.

Solution: Let u and v denote two C^2 solutions of the initial–boundary–value problem (23), and put $w = u - v$. Then, by the linearity of the PDE and the conditions in (23), w is a solution of the initial–boundary–value problem (29). By the result of part (b), $w(x,t) = 0$ for all $x \in [0, L]$ and all $t \geqslant 0$, so that $u = v$. Thus, the initial–boundary–value problem (23) can have at most one solution. □

5. In this problem we consider the following nonlinear boundary value problem:

$$\begin{cases} -\Delta u &= g(u), \quad \text{in } R; \\ u &= 0, \quad \text{on } \partial R, \end{cases} \tag{30}$$

where R is a bounded open subset of $\mathbb{R}^3$ with smooth boundary, ∂R, and

$$g\colon \mathbb{R} \to \mathbb{R}$$

is a continuous real valued function of a single real variable. Define $G\colon \mathbb{R} \to \mathbb{R}$ by

$$G(\xi) = \int_0^\xi g(s)\, ds, \quad \text{for all } \xi \in \mathbb{R}. \tag{31}$$

Denote by $C_o^2(R)$ the space of functions $\{u \in C^2(R) \cap C(\overline{R}) \mid u = 0 \text{ on } \partial R\}$; that is, $C_o^2(R)$ is the space of C^2 functions in R that vanish on the boundary of R.

Define the functional $J\colon C_o^2(R) \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\iiint_R |\nabla u|^2\, dV - \iiint_R G(u)\, dV, \quad \text{for all } u \in C_o^2(R). \tag{32}$$

(a) For given $u \in C_o^2(R)$ and $\varphi \in C_c^\infty(R)$, define $h\colon \mathbb{R} \to \mathbb{R}$ by

$$h(t) = J(u + t\varphi), \quad \text{for } t \in \mathbb{R}. \tag{33}$$

Compute $h'(t)$ for all t in $\mathbb{R}$ and show that

$$h'(0) = \iiint_R \nabla u \cdot \nabla\varphi\, dV - \iiint_R g(u)\varphi\, dV.$$

***Solution**:* Compute h in (33) using (32) to get

$$\begin{aligned} h(t) &= \frac{1}{2}\iiint_R |\nabla(u + t\varphi)|^2\, dV - \iiint_R G(u + t\varphi)\, dV \\ &= \frac{1}{2}\iiint_R |\nabla u + t\nabla\varphi|^2\, dV - \iiint_R G(u + t\varphi)\, dV \\ &= \frac{1}{2}\iiint_R (\nabla u + t\nabla\varphi)\cdot(\nabla u + t\nabla\varphi)\, dV \\ &\qquad - \iiint_R G(u + t\varphi)\, dV \\ &= \frac{1}{2}\iiint_R [|\nabla u|^2 + 2t\nabla u\cdot\nabla\varphi + t^2|\nabla\varphi|^2]\, dV \\ &\qquad - \iiint_R G(u + t\varphi)\, dV, \end{aligned}$$

so that, using (32),

$$\begin{aligned} h(t) &= J(u) + t \iiint_R \nabla u \cdot \nabla\varphi \; dV + \frac{t^2}{2} \iiint_R |\nabla\varphi|^2 \; dV \\ &\quad - \iiint_R G(u + t\varphi) \; dV + \iiint_R G(u) \; dV, \end{aligned} \tag{34}$$

for all t.

Next, differentiate on both sides of (34) with respect to t to get

$$\begin{aligned} h'(t) &= \iiint_R \nabla u \cdot \nabla\varphi \; dV + t \iiint_R |\nabla\varphi|^2 \; dV \\ &\quad - \frac{d}{dt}\left[\iiint_R G(u + t\varphi) \; dV \right], \end{aligned} \tag{35}$$

for all t, where, differentiating under the integral sign and using the Chain Rule,

$$\begin{aligned} \frac{d}{dt}\left[\iiint_R G(u + t\varphi) dV \right] &= \iiint_R \frac{\partial}{\partial t}[G(u + t\varphi)] \; dV \\ &= \iiint_R G'(u + t\varphi)\varphi \; dV \end{aligned}$$

so that, by virtue of (31) and the Fundamental Theorem of Calculus,

$$\frac{d}{dt}\left[\iiint_R G(u + t\varphi) dV \right] = \iiint_R g(u + t\varphi)\varphi \; dV, \quad \text{for all } t. \tag{36}$$

Substituting the result of (36) into the right–hand side of (35) then yields

$$\begin{aligned} h'(t) &= \iiint_R \nabla u \cdot \nabla\varphi \; dV + t \iiint_R |\nabla\varphi|^2 \; dV \\ &\quad - \iiint_R g(u + t\varphi)\varphi \; dV, \end{aligned} \tag{37}$$

for all t. Thus, substituting $t = 0$ in (37) then yields

$$h'(0) = \iiint_R \nabla u \cdot \nabla\varphi \; dV - \iiint_R g(u)\varphi \; dV, \tag{38}$$

which was to be shown. □

(b) Show that if u is a minimizer of the functional J defined in (32) in the space $C_o^2(R)$, then

$$\iiint_R \nabla u \cdot \nabla \varphi \, dV - \iiint_R g(u)\varphi \, dV = 0, \quad \text{for all } \varphi \in C_c^\infty(R). \tag{39}$$

***Solution**:* Suppose that u is a minimizer of J in $C_o^2(R)$. Thus, for any $\varphi \in C_c^\infty(R)$,

$$J(u) \leqslant J(u + t\varphi), \quad \text{for all } t \in \mathbb{R}, \tag{40}$$

since $u + t\varphi \in C_o^2(R)$ for all $\varphi \in C_c^\infty(R)$. It then follows from (32), (33) and (40) that

$$h(0) \leqslant h(t), \quad \text{for all } t \in \mathbb{R}. \tag{41}$$

It follows from (41) that h has a minimum at 0; thus, since h is differentiable, $h'(0) = 0$. Hence, in view of (38),

$$\iiint_R \nabla u \cdot \nabla \varphi \, dV - \iiint_R g(u)\varphi \, dV = 0, \quad \text{for all } \varphi \in C_c^\infty(R),$$

which is (39). □

(c) Show that, if (39) holds true for $u \in C_o^2(R)$, then u is a solution of the BVP in (30).

***Solution**:* Suppose that (39) holds true for $u \in C_o^2(R)$. Then, integrating by parts,

$$-\iiint_R \Delta u\varphi \, dV - \iiint_R g(u)\varphi \, dV = 0, \quad \text{for all } \varphi \in C_c^\infty(R),$$

where we have used the fact that φ vanishes in a neighborhood of ∂R for all $\varphi \in C_c^\infty(R)$. We then have that

$$\iiint_R [\Delta u + g(u)]\varphi \, dV = 0, \quad \text{for all } \varphi \in C_c^\infty(R). \tag{42}$$

It follows from (42) and the result of Problem 2 in Assignment #6 that

$$\Delta u + g(u) = 0 \quad \text{in } R,$$

since we are assuming that $u \in C^2(R)$ and $g \colon \mathbb{R} \to \mathbb{R}$ is continuous. Thus, u solves the PDE in (30). Since we are also assuming that $u \in C_o^2(R)$, u also satisfies the boundary condition in (30). Hence, u is a solution of the BVP in (30). □