Exam 2

Due on Friday, April 25, 2014

Name: __________________________________

This is an open–notes, open text exam; you may consult your own notes, or the class notes in my courses website at `http://pages.pomona.edu/~ajr04747/`, or the text for the course.

Students are expected to work individually on these problems. You may not consult with anyone.

Show all significant work and provide reasoning for all your assertions.

Write your name on this page and staple it to your solutions. Turn in your solutions at the start of class on **Friday, April 25, 2014.**

I have read and agree to these instructions. Signature: ________________________

1. Construct a solution to the following initial–boundary value problem for the one–dimensional wave equation on the half–line:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0, & \text{for } x > 0, t > 0; \\ u(0,t) = 0, & \text{for } t > 0; \\ u(x,0) = f(x), & \text{for } x > 0; \\ u_t(x,0) = g(x), & \text{for } x > 0, \end{cases}$$

   where f and g are given continuous functions of a single variable.

2. Let $D_a(x_o, y_o)$ denote the open disk of radius $a > 0$ around the point (x_o, y_o) in the xy–plane; that is,

$$D_a(x_o, y_o) = \{(x, y) \in \mathbb{R}^2 \mid (x - x_o)^2 + (y - y_o)^2 < a^2\}.$$

   Use the Poisson kernel for the open unit disk, D_1, derived in class to construct a solution to the Dirichlet problem for $D_a(x_o, y_o)$:

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } D_a(x_o, y_o); \\ u(x,y) = f(x,y), & \text{for } (x,y) \in \partial D_a(x_o, y_o), \end{cases}$$

   where f is a given continuous function defined and continuous on a neighborhood of the boundary of $D_a(x_o, y_o)$.

3. The one–dimensional heat equation

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0, \qquad \text{for } 0 < x < L, t > 0, \tag{1}$$

models the flow of heat in a cylindrical rod of length L, constant–cross sectional area, and thermal diffusivity k. The value $u(x,t)$ gives the temperature in the cross–section of the rod at x and time t. In this problem we construct a solution to the initial–boundary–value problem for the PDE in (1):

$$\begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial^2 u}{\partial x^2} = 0, & \text{for } 0 < x < L, t > 0, \\ u(x,0) = f(x), & \text{for all } x \in [0,L]; \\ u_x(0,t) = 0, & \text{for all } t; \\ u_x(L,t) = 0, & \text{for all } t, \end{cases} \tag{2}$$

where $f\colon [0,L] \to \mathbb{R}$ is a given continuous function.

(a) Give an interpretation for the boundary conditions in (2).

(b) Use separation of variables and the principle of superposition to construct a solution to initial–boundary value problem in (2). State conditions that the initial temperature distribution, f, must satisfy in order for your construction to be valid.

(c) What does the solution, u, constructed in part (a) predict about the long–term behavior of the temperature distribution, $u(x,t)$, as $t \to \infty$? Provide an interpretation.

4. In class and in the lecture notes we derived the one–dimensional heat kernel,

$$K(x,t) = \frac{1}{\sqrt{4\pi Dt}}\, e^{-x^2/4Dt}, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0.$$

Given a bounded and continuous function $g\colon \mathbb{R} \times [0,\infty) \to \mathbb{R}$, define

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} K(x-\xi, t-s) g(\xi, s)\; d\xi ds, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0.$$

Verify that u is a solution for the following initial–value problem for the non-homogeneous one–dimensional heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\dfrac{\partial^2 u}{\partial x^2} = g(x,t), & x \in \mathbb{R},\ t > 0; \\ u(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$