Assignment #8

Due on Friday, February 27, 2015

Read Section 4.2 on *Analysis of Linear Systems* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read Chapter 3, on *Linear Systems,* in Blanchard, Devaney and Hall.
Background and Definitions.
Linearly Independent Functions.
Let $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}$ and $\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$, for $t \in I$, where I denotes an open interval of real numbers, define vector valued functions over I. We say that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are linearly independent if

$$\det \begin{pmatrix} x_1(t_o) & x_2(t_o) \\ y_1(t_o) & y_2(t_o) \end{pmatrix} \neq 0, \qquad \text{for some } t_o \in I. \tag{1}$$

The determinant on the left–hand side of (1) is called the Wronskian of the functions $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, and we will denote it by $W(t)$.

Do the following problems

1. Let v_1 and v_2 be vectors in $\mathbb{R}^2$ that are linearly independent. Define

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = e^{\lambda_1 t}\mathrm{v}_1 \quad \text{and} \quad \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = e^{\lambda_2 t}\mathrm{v}_2, \quad \text{for } t \in \mathbb{R},$$

   where λ_1 and λ_2 are real numbers. Show that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are linearly independent.

2. Define

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} \cos(\beta t) \\ \sin(\beta t) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} -\sin(\beta t) \\ \cos(\beta t) \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

   where β is a nonzero real number. Show that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are linearly independent.

3. Define

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda t} \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} te^{\lambda t} \\ e^{\lambda t} \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

where λ is a real number. Show that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are linearly independent.

4. Consider the general linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A(t) \begin{pmatrix} x \\ y \end{pmatrix}, \tag{2}$$

where

$$A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, \quad \text{for } t \in I, \tag{3}$$

and a, b, c and d are continuous functions defined in some open interval I.

Let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be two solutions of (2) and let $W(t)$ denote their Wronskian.

Verify that

$$\frac{dW}{dt} = p(t)W, \quad \text{for } t \in I,$$

where $p(t) = \text{trace}(A(t))$, for all $t \in I$; that is, $p(t)$ is the trace of the matrix $A(t)$ in (3).

5. *(Problem 4 Continued).* Let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be two solutions of (2) and let $W(t)$ denote their Wronskian.

Show that, if $W(t_o) \neq 0$ for some $t_o \in I$, then $W(t) \neq 0$ for all $t \in I$.