Exam 1 (Part II)

Due on Friday, February 20, 2015

Name: ____________________________

This is the out–of–class portion of Exam 1. There are three questions in this portion of the exam. This is a closed–book and closed–notes exam. Show all significant work and give reasons for all your answers.

Students are expected to work individually on these problems. You may not consult with anyone.

Please, write your name on this page and staple it to your solutions. Turn in your solutions at the start of class on **Friday, February 20, 2015.**

I have read and agree to these instructions. Signature: ____________________

1. In Problem 5 of Assignment #1, you derived the logistic model for population growth

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{L}\right), \tag{1}$$

where r is a positive constant called the intrinsic growth rate, and L is a positive constant known as the carrying capacity. $N(t)$ denotes the bacterial density in a culture at time t. Assume that the initial bacterial density is $N(0) = N_o$, for some constant $N_o > 0$.

(a) Assume that N is a solution of (1) and that $N(t) \neq 0$ for all t. Make the change of variables

$$u(t) = \frac{1}{N(t)}, \qquad \text{for all } t.$$

Derive a differential equation that the function u must satisfy. What type of differential equation do you obtain?

(b) Construct solutions to the differential equation that you derived in part (b), and compute $\lim_{t\to\infty} u(t)$.

(c) Give a solution to the differential equation that you derived in part (b) subject to the initial condition $u(0) = \dfrac{1}{N_o}$.

(d) Use the result from part (c) to construct a solution to the logistic equation in (1) subject to the initial condition $N(0) = N_o$. What does this solution predict the limiting value of $N(t)$ will be as $t \to \infty$?

2. In this problem, you will construct solutions to the second order differential equation

$$\frac{d^2x}{dt^2} - x = 0. \tag{2}$$

   (a) Turn the equation in (2) into a two–dimensional system of first order equations, and construct solutions to the system that you obtain.

   (b) Use your result from part (a) to obtain solutions to the differential equation in (2).

   (c) Give a solution to (2) subject to the initial conditions $x(0) = 1$ and $x'(0) = 0$.

3. Consider the two–dimensional linear system

$$\begin{cases} \dot{x} &= 3x - y; \\ \dot{y} &= x + y. \end{cases} \tag{3}$$

   (a) Construct solutions to the system in (3).

   (b) Sketch the nullclines of the system in (3) and indicate the directions of the tangents to the trajectories on them.

   (c) Use the solutions you got in part (a) and the information gained in part (b) to sketch the phase portrait of the system in (3).