Math 102. Rumbos Spring 2010

Solutions to Exam 1 (Part II).

(1) Solve the logistic equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{L}\right), \qquad (1)$$

(a) Set $u(t) = \dfrac{1}{N(t)}$, for all t with $N(t) \neq 0$, and use the Chain Rule to compute

$$\frac{du}{dt} = -\frac{1}{N^2}\frac{dN}{dt}$$

$$= -\frac{1}{N^2}\left[rN\left(1 - \frac{N}{L}\right)\right], \text{ by (1)}$$

$$= -\frac{r}{N}\left(1 - \frac{N}{L}\right)$$

$$= -r\frac{1}{N} + \frac{r}{L};$$

so that $\dfrac{du}{dt} = -ru + \dfrac{r}{L}$, which is a first-order, linear ODE.

(b) We can solve $\dfrac{du}{dt} = -ru + \dfrac{r}{L}$ by separating variables, or by using an integrating factor to get

$$u(t) = \frac{1}{L} + Ce^{-rt}$$

for constant of integration, C.

Note that, since $r > 0$, $\lim_{t \to \infty} u(t) = \dfrac{1}{L}$.

(c) If $u(0) = \frac{1}{N_0}$, we obtain from

$$u(t) = \frac{1}{L} + c\, e^{-rt}$$

that $\frac{1}{L} + c = \frac{1}{N_0}$ $\quad c = \frac{1}{N_0} - \frac{1}{L}$,

or $c = \frac{L - N_0}{N_0 L}$

Thus, $u(t) = \frac{1}{L} + \frac{L-N_0}{N_0 L} e^{-rt}$, $t \in \mathbb{R}$.

(d) From $u(t) = \frac{1}{N(t)}$, for t with $N(t) \neq 0$, we obtain that

$$N(t) = \frac{1}{u(t)} = \frac{1}{\frac{1}{L} + \frac{L-N_0}{N_0 L} e^{-rt}}, \text{ for all } t$$

or $$N(t) = \frac{N_0 L}{N_0 + (L - N_0) e^{-rt}}$$

Note that $\lim_{t\to\infty} N(t) = \frac{N_0 L}{N_0} = L$,

since $r > 0$. Thus, the logistic model predicts that, if $N(0) > 0$, then $\lim_{t\to\infty} N(t) = L$.

2 (a) Let $y = \frac{dx}{dt}$; so that $\dot{x} = y$ and $\dot{y} = \frac{d^2x}{dt^2} = x$, according to (2). We then have that the second order ODE in (2) is equivalent to the two-dimensional system

$$\begin{cases} \dot{x} = y \\ \dot{y} = x \end{cases}$$

Write the system in vector form as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

or $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Put $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix A has characteristic polynomial $p_A(\lambda) = \lambda^2 - 1$, which can be factored as $p_A(\lambda) = (\lambda+1)(\lambda-1)$. Therefore, A has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$.

We compute corresponding eigenvectors to get $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Therefore, the general solution of the system is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-t} v_1 + c_2 e^{t} v_2$

or $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} + c_2 e^{t} \\ -c_1 e^{-t} + c_2 e^{t} \end{pmatrix}$, for all $t \in \mathbb{R}$.

(b) Picking out the first component of the solution of the system in part (a), we obtain that

$$x(t) = c_1 e^{-t} + c_2 e^{t}, \quad t \in \mathbb{R}$$

solves (2).

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(c) $x(0) = 1 \Rightarrow c_1 + c_2 = 1$

$x'(0) = 0 \Rightarrow -c_1 + c_2 = 0.$

Solving this system of equations we obtain that $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$.

Therefore,

$$x(t) = \frac{1}{2} e^{-t} + \frac{1}{2} e^{t}, \quad \text{for } t \in \mathbb{R},$$

solves the IVP $\begin{cases} \ddot{x} - x = 0 \\ x(0) = 1 \\ x'(0) = 0. \end{cases}$

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3. Write the system in (3) in matrix form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix},$$

where $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$

The characteristic polynomial of A is $p_A(\lambda) = \lambda^2 - 4\lambda + 4$, which factors into $p_A(\lambda) = (\lambda - 2)^2$.

Thus, $\lambda = 2$ is the only eigenvalue of A.

A corresponding eigenvector is given by

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

To construct a solution of the system, we first find a solution of

$$(A - \lambda I)\, v = v_1$$

to get $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Then, set $Q = [\, v_1 \;\; v_2 \,]$ and make the change of variables

$$\begin{pmatrix} u \\ v \end{pmatrix} = Q^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$$

to get the system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = J \begin{pmatrix} u \\ v \end{pmatrix},$$

where $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

or $\begin{cases} \dot{u} = \lambda u + v; \\ \dot{v} = \lambda v, \end{cases}$

which has general solution

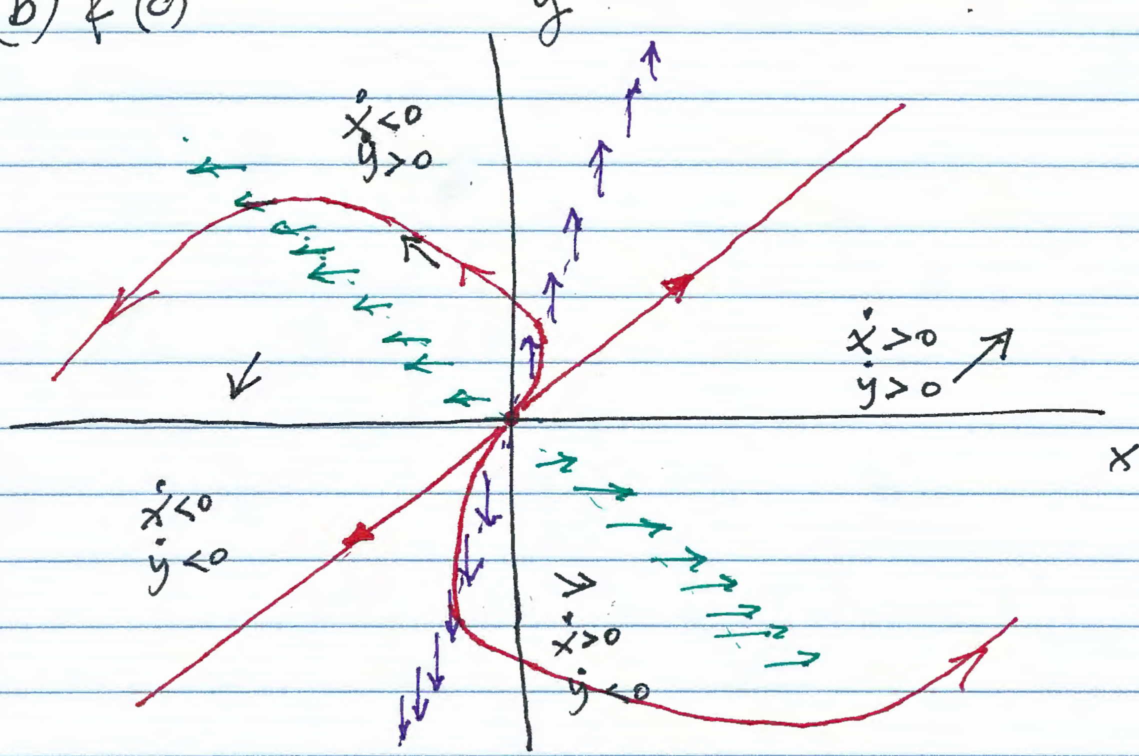
$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} + c_2 t e^{2t} \\ c_2 e^{2t} \end{pmatrix}, \quad \text{for } t \in \mathbb{R}.$$

Thus, the general solution of the system $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$ is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = Q \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = (c_1 e^{2t} + c_2 t e^{2t}) v_1 + c_2 e^{2t} v_2,$$

for all $t \in \mathbb{R}$.

(b) f (c)



$\dot{x}=0$ - nullcline : $y = 3x$

$\dot{y}=0$ - nullcline : $y = -x$