Exam 2 (Part I)

Wednesday, April 1, 2015

Name: ____________________________________

This is the in–class portion of Exam 2. This is a closed–book and closed–notes exam. Show all significant work and give reasons for all your answers. Use your own paper and/or the paper provided by the instructor. You have up to 50 minutes to work on the following 2 questions. Relax.

1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$ denote continuous functions and consider the autonomous, two–dimensional system

$$\begin{cases} \dfrac{dx}{dt} = f(x,y); \\[2ex] \dfrac{dy}{dt} = g(x,y). \end{cases} \tag{1}$$

Let (x_o, y_o) denote a given point in the xy–plane.

(a) State conditions on f and g that will guarantee that the system (1) will have a unique solution satisfying $(x(0), y(0)) = (x_o, y_o)$ and defined on some interval around $t = 0$.

(b) Assume that the system in (1) has a solution that exists for all $t \in \mathbb{R}$ and is given by $(x(t), y(t))$, for all $t \in \mathbb{R}$, where $x\colon \mathbb{R} \to \mathbb{R}$ and $y\colon \mathbb{R} \to \mathbb{R}$ are differentiable functions. For a given $\tau \in \mathbb{R}$, define

$$(u(t), v(t)) = (x(t+\tau), y(t+\tau)), \quad \text{for all } t \in \mathbb{R}.$$

Show that (u, v) is also a solution of the system (1).

(c) Assume that $(\overline{x}, \overline{y})$ is an equilibrium point of the autonomous system in (1). Define $(x, y)\colon \mathbb{R} \to \mathbb{R}^2$ by

$$(x(t), y(t)) = (\overline{x}, \overline{y}), \quad \text{for all } t \in \mathbb{R}.$$

Prove that (x, y) is a solution of the system (1). Explain your reasoning.

2. Sketch the nullclines and find the equilibrium point of the system

$$\begin{cases} \dot{x} = x + y - 1 \\ \dot{y} = -x + y \end{cases}$$

Determine the nature of the stability of the equilibrium point. Sketch the phase portrait.