Math 102. Rumbos Spring 2015

Solutions to Exam 2 (Part I)

(1)(a) If the functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$ have continuous partial derivatives in an open region containing (x_o, y_o), then the IVP

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \\ x(0) = x_o \\ y(0) = y_o \end{cases}$$

is guaranteed to have a unique solution defined on some interval containing $t_o = 0$.

(b) Assume that $(x(t), y(t))$, for $t \in \mathbb{R}$, defines a solution of (1), so that

$$x'(t) = f(x(t), y(t)), \text{ for all } t,$$

and $\quad y'(t) = g(x(t), y(t))$, for all t.

Define

$$u(t) = x(t+\tau), \quad v(t) = y(t+\tau), \text{ for all } t \in \mathbb{R},$$

where τ is a fixed real number. Then, by the Chain Rule

$$u'(t) = x'(t+\tau) \cdot \frac{d}{dt}(t+\tau) = x'(t+\tau);$$

so that

$$u'(t) = f(x(t+\tau), y(t+\tau)) = f(u(t), v(t)),$$

for all t. Similarly,

$$v'(t) = g(u(t), v(t)), \text{ for all } t.$$

Hence, (u,v) is also a solution of (1).

(c) Assume that $(\bar{x}, \bar{y})$ is an equilibrium point of (1), so that

$$f(\bar{x}, \bar{y}) = 0 \text{ and } g(\bar{x}, \bar{y}) = 0.$$

Define $(x, y) : \mathbb{R} \to \mathbb{R}^2$ by

$$(x(t), y(t)) = (\bar{x}, \bar{y}) \text{ for all } t \in \mathbb{R}.$$

We then have that

$$x(t) = \bar{x} \text{ for all } t.$$

Thus, $x'(t) = 0 = f(\bar{x}, \bar{y}) = f(x(t), y(t))$ for all t. Similarly,

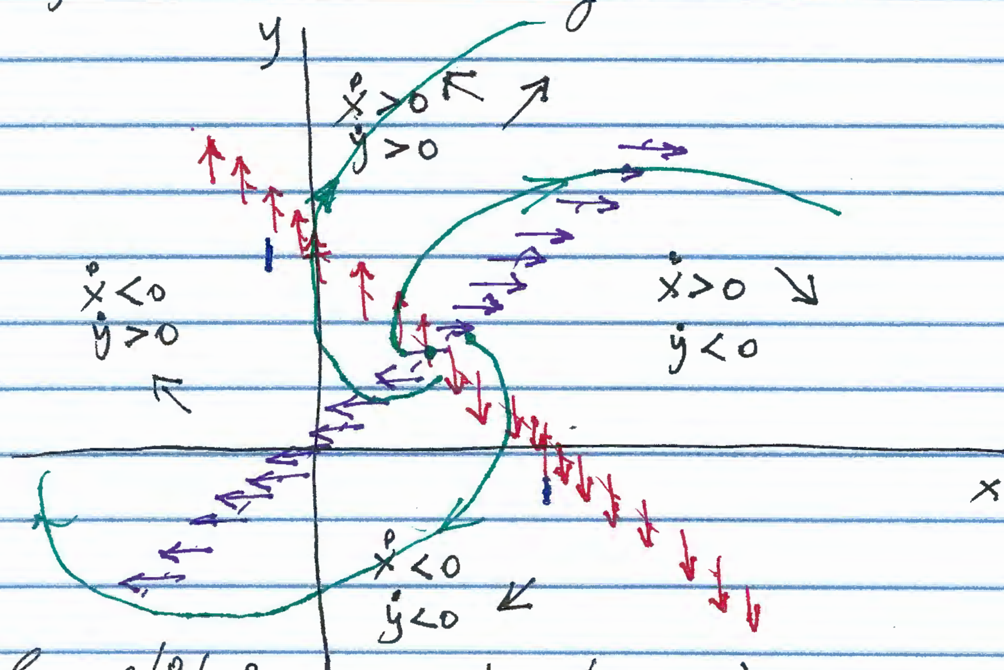
$$y'(t) = g(x(t), y(t)), \text{ for all } t \in \mathbb{R}.$$

Consequently (x, y) solves (1).

2. $\begin{cases} \dot{x} = x + y - 1 \\ \dot{y} = -x + y \end{cases}$

$\dot{x} = 0$-nullcline : $x + y = 1$

$\dot{y} = 0$-nullcline : $y = x$



Equilibrium point: $\left(\frac{1}{2}, \frac{1}{2}\right)$

Make the change of variables

$$u = x - \frac{1}{2} \quad , \quad v = y - \frac{1}{2}$$

Then $\dot{u} = \dot{x} = x + y - 1 = (x - \frac{1}{2}) + (y - \frac{1}{2})$;
so that

$$\dot{u} = u + v.$$

Similarly,

$$\dot{v} = \dot{y} = -x + y$$
$$= -x + \frac{1}{2} + y - \frac{1}{2}$$
$$= -(x - \frac{1}{2}) + (y - \frac{1}{2})$$
$$= -u + v$$

Thus, (u,v) solves the system

$$\begin{cases} \dot{u} = u + v; \\ \dot{v} = -u + v, \end{cases}$$

which is of the form $\begin{cases} \dot{u} = \alpha u - \beta v \\ \dot{v} = \beta u + \alpha v \end{cases}$
where $\alpha = 1$, $\beta = -1$. Hence, since $\alpha > 0$ and $\beta < 0$, the trajectories of $\begin{cases} \dot{u} = u + v \\ \dot{v} = -u + v \end{cases}$ will spiral outward from $(\bar{u}, \bar{v}) = (0, 0)$ in the clockwise sense. Consequently, the trajectories of $\begin{cases} \dot{x} = x + y - 1 \\ \dot{y} = -x + y \end{cases}$ will spiral outward around $(\frac{1}{2}, \frac{1}{2})$ in the clockwise sense.