Math 102. Rumbos Spring 2015

Solutions to Exam 2 (Part II)

(1) Write the system in matrix form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix},$$

where $A = \begin{pmatrix} 6 & 4 \\ -10 & -6 \end{pmatrix}$.

The characteristic polynomial of A is

$$p_A(\lambda) = \lambda^2 + 4,$$

which has eigenvalues $\lambda = \pm 2i$.
We find a matrix $Q = [v_1 \ v_2]$ such that

$$Q^{-1} A Q = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

First, we find an eigenvector $w_1 \in \mathbb{C}^2$ corresponding to $\lambda_1 = 2i$ by solving

$$(A - \lambda_1 I) w = \mathbf{0}$$

or

$$\begin{pmatrix} 6-2i & 4 \\ -10 & -6-2i \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

Using Gaussian elimination to solve this system, we get

$$\left(\begin{array}{cc|c} 3-i & 2 & 0 \\ -5 & -3-i & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -5 & -3-i & 0 \\ 3-i & 2 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & \frac{3+i}{5} & 0 \\ 3-i & 2 & 0 \end{array} \right)$$

Thus, multiplying the first row by $-(3-i)$ and adding to the second row yields

$$\left(\begin{array}{cc|c} 1 & \frac{3+i}{5} & 0 \\ 0 & 0 & 0 \end{array} \right),$$

since

$$-(3-i)\frac{(3+i)}{5} + 2 = -\frac{(3^2-i^2)}{5} + 2$$

$$= -\frac{10}{5} + 2$$

$$= 0$$

Thus, the system $(A-\lambda_1 I)w = 0$ is equivalent to the equation

$$z_1 + \frac{3+i}{5} z_2 = 0,$$

which can be solved for z_1 to yield

$$z_1 = -\frac{3+i}{5} z_2$$

Set $z_2 = -5$. Then, $z_1 = 3+i$

Thus, $w_1 = \begin{pmatrix} 3+i \\ -5 \end{pmatrix}$

is an eigenvector corresponding to $\lambda_1 = 2i$

Put

$$v_1 = \text{Im}(w_1) \quad \text{and} \quad v_2 = \text{Re}(w_1),$$

or

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

Then, $Q = \begin{pmatrix} 1 & 3 \\ 0 & -5 \end{pmatrix}$; so that

$$Q^{-1} = \begin{pmatrix} 1 & 3/5 \\ 0 & -1/5 \end{pmatrix}$$

and $Q^{-1} A Q = \begin{pmatrix} 1 & 3/5 \\ 0 & -1/5 \end{pmatrix} \begin{pmatrix} 6 & 4 \\ -10 & -6 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & -5 \end{pmatrix}$

$$= \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

Setting $J = Q^{-1} A Q$, we compute the fundamental matrix of J to get

$$E_J(t) = \begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix}.$$

Then, the fundamental matrix of A is

$$E_A(t) = Q\, E_J(t)\, Q^{-1}$$

$$= \begin{pmatrix} 1 & 3 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix} \begin{pmatrix} 1 & 3/5 \\ 0 & -1/5 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2t) + 3\sin(2t) & 2\sin 2t \\ -5\sin(2t) & -3\sin(2t) + \cos(2t) \end{pmatrix}$$

for all t.

Thus, the general solution of $\begin{cases} \dot{x} = 6x + 4y \\ \dot{y} = -10x - 6y \end{cases}$

is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = E_A(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, for all t,

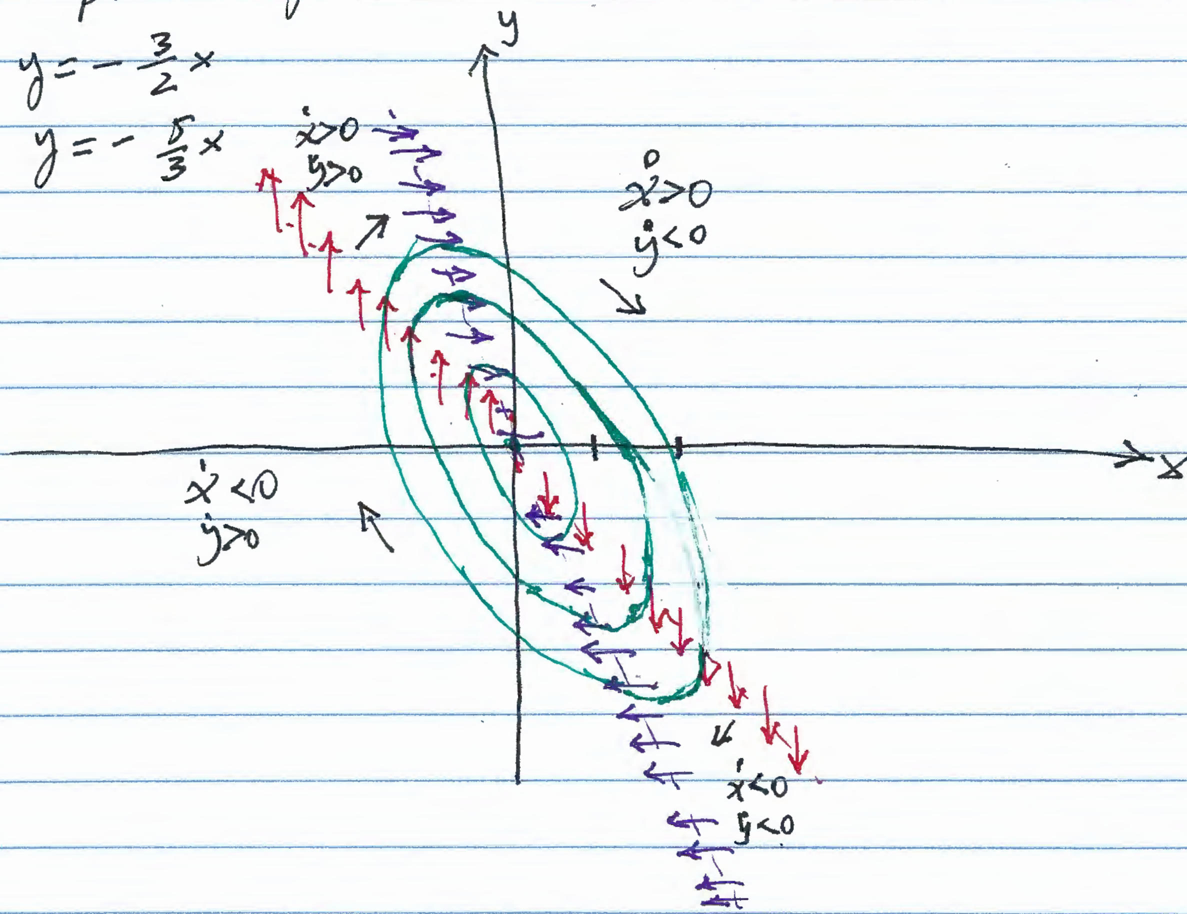
or $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 \cos(2t) + 3c_1 \sin(2t) + 2c_2 \sin 2t \\ -5c_1 \sin(2t) - 3c_2 \sin(2t) + c_2 \cos(2t) \end{pmatrix}$,

for all $t \in \mathbb{R}$.

(b) The origin is a center. The trajectories are concentric ellipses about the origin. We use nullclines to help sketch the phase-portrait

$\dot{x} = 0$-nullcline: $y = -\frac{3}{2}x$

$\dot{y} = 0$-nullcline: $y = -\frac{5}{3}x$



2. Write the system on the form

$$\begin{cases} \dot{x} = x(1-2x-0.5y) \\ \dot{y} = 0.5y(x-1) \end{cases}$$

(a) If $x=0$, $\frac{dy}{dt} = -0.5y$; so that the population of density y will decay exponentially to 0 (extinction).

If $y=0$, $\frac{dx}{dt} = x\left(1-\frac{x}{1/2}\right)$ (the logistic equation with intrinsic growth rate 1 and carrying capacity 1/2). Thus, in the absence of the population of density y, the population of density x will experience logistic growth.

If both species are present, the presence of y has a negative effect on the per-capita growth rate of the population of density x:

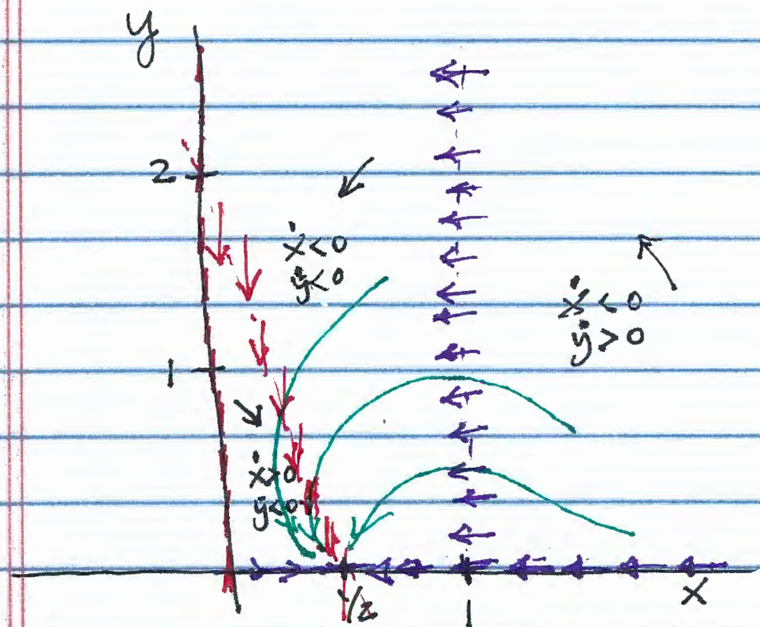
$$\frac{1}{x}\frac{dx}{dt} = 1-2x-0.5y.$$

On the other hand, the presence of x has a positive effect on the per-capita growth rate of the species of density y:

$$\frac{1}{y}\frac{dy}{dt} = 0.5x - 0.5$$

Thus, the system models a predator-prey interaction.

(b) The $\dot{x}=0$-nullclines are the lines
$x=0$ (the y-axis) and $4x+y=2$.
The $\dot{y}=0$-nullclines are the lines
$y=0$ (the x-axis) and $x=1$
These are sketched below.



Equilibrium points on the first quadrant:
$(0,0)$ & $(1/2, 0)$

To determine the nature of the stability of the equilibrium points, we look at the linearization of field

$$F(x,y) = \begin{pmatrix} x - 2x^2 - 0.5xy \\ 0.5xy - 0.5y \end{pmatrix};$$

namely, $DF(x,y) = \begin{pmatrix} 1-4x-0.5y & -0.5x \\ 0.5y & 0.5x-0.5 \end{pmatrix}$,

at the equilibrium points.
At $(0,0)$ we get
$$DF(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -0.5 \end{pmatrix},$$
which has eigenvalues $\lambda_1 = -0.5 < 0$ and $\lambda_2 = 1 > 0$.

Thus, $(0,0)$ is a saddle point.

At $(1/2, 0)$ we get

$$DF(1/2, 0) = \begin{pmatrix} -1 & -\frac{1}{4} \\ 0 & -1/4 \end{pmatrix},$$

which has eigenvalues $\lambda_1 = -\frac{1}{4} < 0$ and $\lambda_2 = -1 < 0$. Hence, $(1/2, 0)$ is a sink.

—o—

(c) The model predicts that any trajectory that starts on the positive portion of the first quadrant will tend towards the sink $(1/2, 0)$ as $t \to \infty$; that is,

$$\lim_{t\to\infty} x(t) = \frac{1}{2} \quad \text{and} \quad \lim_{t\to\infty} y(t) = 0.$$

Hence, the predator population will not survive on the long run.

—o—

3. (a) Write the 2nd order ode

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + \alpha x = 0$$

as a 2-dimensional system by setting $y = \frac{dx}{dt}$, so that

$$\dot{x} = y$$

and

$$\dot{y} = \frac{d^2x}{dt^2} = -2\frac{dx}{dt} - 2x = -2x - 2y.$$

Hence, the given 2nd order equation is equivalent to the system

$$\begin{cases} \dot{x} = y; \\ \dot{y} = -2x - 2y, \end{cases}$$

which we can write in vector form as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix},$$

where $A = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}$.

The characteristic polynomial of A is

$$\begin{aligned} p_A(\lambda) &= \lambda^2 + 2\lambda + 2 \\ &= (\lambda^2 + 2\lambda + 1) + 1 \\ &= (\lambda + 1)^2 + 1; \end{aligned}$$

so that, the zeros of $p_A(\lambda)$ are

$\lambda = -1 \pm i$

Hence, the fundamental matrix of A, $E_A(t)$, is of the form

$$E_A(t) = Q\, E_J(t)\, Q^{-1},$$

for some invertible matrix Q, where

$$E_J(t) = e^{-t} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Thus, the first component of

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = E_A(t) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$ for

any initial points (x_0, y_0) is a linear combination of the functions

$x_1(t) = e^{-t}\cos t$ and $x_2(t) = e^{-t}\sin t$, for $t \in \mathbb{R}$. Therefore, the general solution of the given 2nd order ODE is

$$x(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t, \text{ for } t \in \mathbb{R},$$

and arbitrary constants c_1 & c_2

(b) $x(0) = 1 \Rightarrow c_1 = 1$;

so, $x(t) = e^{-t}\cos t + c_2 e^{-t}\sin t$, for all t.

Next, compute

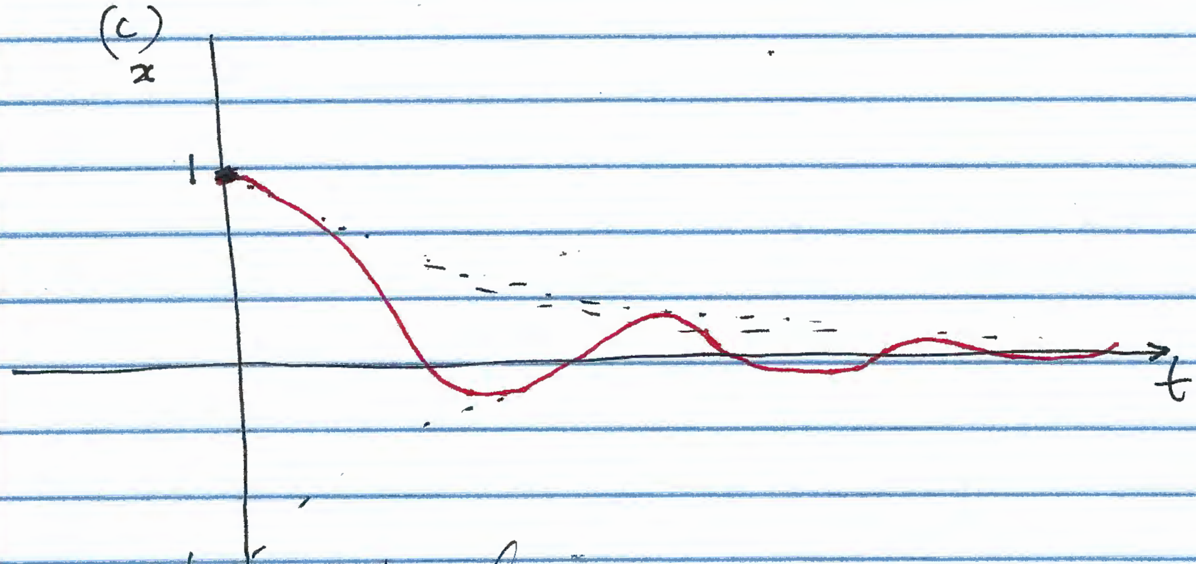
$$x'(t) = -e^{-t}\cos t - e^{-t}\sin t + c_2 e^{-t}\cos t - c_2 e^{-t}\sin t, \text{ for } t \in \mathbb{R}.$$

$x'(0) = 0 \Rightarrow -1 + c_2 = 0$, so that $c_2 = 1$ and therefore

$$x(t) = e^{-t}\cos t + e^{-t}\sin t$$

or $x(t) = e^{-t}(\cos t + \sin t)$

(c)



Note that $\lim_{t \to \infty} x(t) = 0$, since $e^{-t} \to 0$ as $t \to \infty$.