Review Problems for Exam 2

1. Compute the fundamental matrix for the system

$$\begin{cases} \dot{x} = -3x - y; \\ \dot{y} = 4x - 3y \end{cases}$$

   Determine the nature of the stability of the equilibrium point $(0,0)$.

2. Compute the general solution of the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

   and describe the nature of the stability of its equilibrium point. Sketch the phase portrait.

3. Find the equilibrium point of the system

$$\begin{cases} \dot{x} = 2x + y + 1; \\ \dot{y} = x - 2y - 1, \end{cases}$$

   and determine the nature of the stability of the point. Sketch the phase portrait.

4. Let A denote a 2×2 matrix satisfying $\det A < 0$.

   (a) Explain why the origin is an isolated equilibrium point of the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}. \tag{1}$$

   (b) Determine the nature of the stability or unstability of the origin for the system in (1). Explain your reasoning.

5. Find two distinct solutions of the initial value problem $\begin{cases} \dot{x} = 6tx^{2/3}; \\ x(0) = 0. \end{cases}$

   Why doesn't this violate the uniqueness assertion of the local existence and uniqueness theorem?

6. Consider the initial value problem $\begin{cases} \dfrac{dy}{dt} = y^2 - y; \\ y(0) = 2 \end{cases}$

   Give the maximal interval of existence for the solution. Does the solution exist for all t? If not, explain what prevents the solution from being extended further.

7. The motion of an object of mass m, attached to a spring of stiffness constant k, and moving along a horizontal flat surface is modeled by the second–order, linear differential equation

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0, \tag{2}$$

where $x = x(t)$ denotes the position of the object along its direction of motion, and γ is the coefficient of friction between the object and the surface.

   (a) Express the equation in (2) as a system of first order linear differential equations:
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3}$$

   (b) For the matrix A in (3), let $\omega^2 = \dfrac{k}{m}$ and $b = \dfrac{\gamma}{2m}$.
   Give the characteristic polynomial of the matrix A, and determine when the A has (i) two real and distinct eigenvalues; (ii) only one real eigenvalue; (iii) complex eigenvalues with nonzero imaginary part.

   (c) Describe the behavior of solutions of (2) in case (iii) of part (b)

8. The following system of first order differential equations can be interpreted as describing the interaction of two species with population densities x and y:

$$\begin{cases} \dfrac{dx}{dt} = x(1 - x - y); \\ \\ \dfrac{dy}{dt} = y(0.5 - 0.25y - 0.75x) \end{cases}$$

   (a) What do these equations predict about the population density of each species if the other were not present? What effect do the species have on each other? Describe the kind of interaction that this system models.

   (b) Sketch the nullclines, determine the equilibrium points, sketch some possible trajectories, and determine the nature of the stability of all the equilibrium points.

   (c) Describe the different possible long–run behaviors of x and y as $t \to \infty$, and interpret the result in terms of the populations of the two species.

9. Let Ω denote an open interval of real numbers, and $f\colon \Omega \to \mathbb{R}$ denote a continuous function. Let $x_p\colon \Omega \to \mathbb{R}$ denote a particular solution of the nonhomogeneous, second–order equation

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t), \qquad \text{for } t \in \Omega, \tag{4}$$

where b and c are real constants.

   (a) Let $x\colon \Omega \to \mathbb{R}$ denote any solution of (4) and put

   $$u(t) = x(t) - x_p(t), \qquad \text{for } t \in \Omega.$$

   Verify that u solves the homogeneous, second–order equation

   $$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0, \qquad \text{for } t \in \Omega, \tag{5}$$

   (b) Let $x_1\colon \Omega \to \mathbb{R}$ and $x_2\colon \Omega \to \mathbb{R}$ denote linearly independent solutions of the homogenous equation (5). Prove that any solution of the nonhomogeneous equation in (4) must be of the form

   $$x(t) = c_1x_1(t) + c_2x_2(t) + x_p(t), \qquad \text{for all } t \in \Omega,$$

   where c_1 and c_2 are constants.

10. Consider the two–dimensional system

$$\begin{cases} \dot{x} = y + x(x^2 + y^2) \\ \dot{y} = -x + y(x^2 + y^2) \end{cases} \tag{6}$$

   (a) Show that $(0, 0)$ is an isolated critical point of the system in (6).

   (b) Compute solutions to the linearization of the system in (6) around the origin.

   (c) Determine the nature of the stability of the origin for the linearized system.

   (d) Let $r^2 = x^2 + y^2$ and note that $r\dfrac{dr}{dt} = x\dfrac{dx}{dt} + y\dfrac{dy}{dt}$.
   Show that the solutions to the system in (6) with initial condition $r(0) = r_o > 0$ becomes unbounded as $t \to 1/2r_o^2$, and hence the equilibrium point $(0, 0)$ for the system in (6) is unstable.

   (e) Explain why (c) and (d) together do not contradict the Principle of Linearized Stability.