Exam 3 (Part I)

Wednesday, April 29, 2015

Name: ______________________________

This is the in–class portion of Exam 3. This is a closed–book and closed–notes exam. Show all significant work and give reasons for all your answers. Use your own paper and/or the paper provided by the instructor. You have up to 50 minutes to work on the following 2 questions. Relax.

1. Consider the population model described by the differential equation
$$\frac{dN}{dt} = aN^2 - bN, \tag{1}$$
where a and b are positive parameters.

   (a) Give the units of the parameters a and b.

   (b) Introduce dimensionless variables $u = \dfrac{N}{\mu}$ and $\tau = \dfrac{t}{\lambda}$ to write the equation in (1) in the dimensionless form
$$\frac{du}{d\tau} = f(u). \tag{2}$$
   Express the scaling parameters μ and λ in terms of the original parameters a and b.

   (c) Sketch the graph of f versus u for positive values of u, find the equilibrium points of the equation in (2) and use Principle of Linearized Stability (when applicable) to determine the nature of the stability of the equilibrium points.

   (d) Sketch the shape of possible solution curves of the equation (2) in the τu–plane for various initial values.

   (e) Explain why the value $\overline{N} = b/a$ is called a **threshold population** value.

2. Give the definition of a conserved quantity for a general, two–dimensional, autonomous system of first–order ODEs and verify that the system
$$\begin{cases} \dot{x} = y; \\ \dot{y} = -x, \end{cases} \tag{3}$$
has conserved quantity $H\colon \mathbb{R}^2 \to \mathbb{R}$. Compute H and use its level sets to help you sketch the phase portrait of the system in (3). Explain your reasoning.