Solutions to Exam 3 (Part I)

1. Consider the population model described by the differential equation

$$\frac{dN}{dt} = aN^2 - bN, \tag{1}$$

where a and b are positive parameters.

(a) Give the units of the parameters a and b.

Answer: The parameter a has units of 1/(time × #individuals), while b has units of 1/time. □

(b) Introduce dimensionless variables

$$u = \frac{N}{\mu} \quad \text{and} \quad \tau = \frac{t}{\lambda} \tag{2}$$

to write the equation in (1) in the dimensionless form

$$\frac{du}{d\tau} = f(u). \tag{3}$$

Express the scaling parameters μ and λ in terms of the original parameters a and b.

Solution: Use the expressions in (2) and the Chain Rule to obtain

$$\begin{aligned} \frac{du}{d\tau} &= \frac{du}{dt} \cdot \frac{dt}{d\tau} \\ &= \lambda \frac{d}{dt}\left[\frac{N}{\mu}\right]; \end{aligned}$$

so that,

$$\frac{du}{d\tau} = \frac{\lambda}{\mu} \cdot \frac{dN}{dt} \tag{4}$$

Substituting the expression for $\dfrac{dN}{dt}$ in (1) into the right–hand side of the equation in (4) then yields

$$\begin{aligned} \frac{du}{d\tau} &= \frac{\lambda}{\mu}\left[aN^2 - bN\right] \\ &= \lambda a u N - \lambda b u, \end{aligned}$$

where we have used the first expression in (2); thus, using (2) again,

$$\frac{du}{d\tau} = \lambda\mu a u^2 - \lambda b u. \tag{5}$$

There are two dimensionless groupings of the parameters in equation (5):

$$\lambda\mu a \quad \text{and} \quad \lambda b.$$

In order to determine λ and μ, we set both of them equal to 1:

$$\lambda\mu a = 1 \quad \text{and} \quad \lambda b = 1, \tag{6}$$

from which we get

$$\lambda = \frac{1}{b} \tag{7}$$

and

$$\mu = \frac{1}{\lambda a};$$

so that, using (7),

$$\mu = \frac{b}{a}. \tag{8}$$

Substituting the expressions in (6) into (5) then yields

$$\frac{du}{d\tau} = u^2 - u,$$

or

$$\frac{du}{d\tau} = u(u-1). \tag{9}$$

Thus, scaling the variables N and t as in (2), respectively, where μ and λ are given by (8) and (7), respectively, allows us to write the equation in (1) in the dimensionless form in (9). Setting

$$f(u) = u(u-1), \quad \text{for } u \in \mathbb{R}, \tag{10}$$

we obtain the form in (3). □

(c) Sketch the graph of f versus u for positive values of u, find the equilibrium points of the equation in (3) and use Principle of Linearized Stability (when applicable) to determine the nature of the stability of the equilibrium points.

Solution: Figure 1 shows a sketch of the graph of f versus u. We see

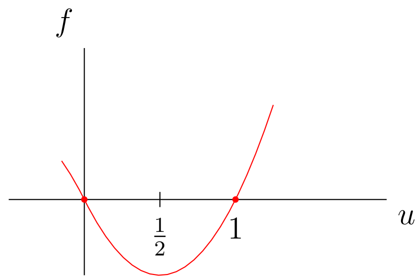

Figure 1: Sketch of graph of f versus u

from the graph that the equation in (3) has two equilibrium points,

$$\overline{u}_1 = 0 \quad \text{and} \quad \overline{u}_2 = 1. \tag{11}$$

We also see from the sketch in Figure 1 that $f'(\overline{u}_1) < 0$; so that, $\overline{u}_1 = 0$ is asymptotically stable, by the Principle of Linearized Stability. Similarly, since $f'(\overline{u}_2) > 0$, $\overline{u}_2 = 1$ is unstable. $\square$

(d) Sketch the shape of possible solution curves of the equation (3) in the tu–plane for various initial values.

***Solution**:* The diagram in Figure 2 summarizes the qualitative information obtained from the graph of f versus u in Figure 1. It follows from the

$f(u)$:	$+$		$-$		$-$		$+$
$f'(u)$:	$-$		$-$		$+$		$+$
		0		$\frac{1}{2}$		1	
u'' :	$-$		$+$		$-$		$+$

Figure 2: Qualitative information of the graph of solutions of (3)

signs in first row in the diagram in Figure 2, and the differential equation in (3) that $u(\tau)$ decreases with increasing τ whenever $0 < u < 1$, and $u(\tau)$ increases with increasing τ whenever $u > 1$.

The third row of signs in the diagram in Figure 2 was obtained by multiplying the signs in the first two rows by virtue of the expression

$$u''(\tau) = f'(u(\tau))f(u(\tau)),$$

obtained by differentiating on both sides of the differential equation in (3) and applying the Chain Rule. We see in the diagram that u'' is positive for $0 < u < 1/2$ or $u > 1$, and u'' is negative for $1/2 < u < 1$. Hence, the shape of the graph of u versus τ is concave up for $0 < u < 1/2$ or $u > 1$, and concave down for $1/2 < u < 1$. Using this information we obtain graphs of solutions for various initial condition shown in Figure 3. $\square$

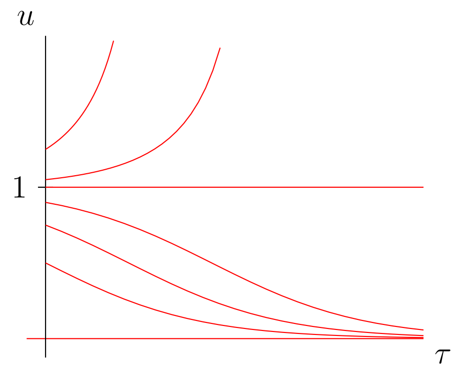

Figure 3: Sketch of Possible Solutions of (3)

(e) Explain why the value $\overline{N} = b/a$ is called a **threshold population** value.

Answer: We see in Figure 3 that the scaled population density, u, will increase without bound if $u(0) > 1$, or tends to 0 as $\tau \to \infty$ if $u(0) < 1$. Thus, $\overline{u}_2 = 1$ is a threshold value between growth and extinction. According to the first expression in (2) and the definition of μ in (8), this value corresponds to a value, $\overline{N}$, of N with

$$\frac{\overline{N}}{b/a} = 1,$$

or

$$\overline{N} = \frac{b}{a}.$$

$\square$

2. Give the definition of a conserved quantity for a general, two–dimensional, autonomous system of first–order ODEs and verify that the system

$$\begin{cases} \dot{x} &= y; \\ \dot{y} &= -x, \end{cases} \tag{12}$$

has a conserved quantity $H\colon \mathbb{R}^2 \to \mathbb{R}$. Compute H and use its level sets to help you sketch the phase portrait of the system in (12). Explain your reasoning.

***Solution**:* A function H of the variables x and y is a conserved quantity for the two dimensional system

$$\begin{cases} \dfrac{dx}{dt} = f(x,y); \\ \\ \dfrac{dy}{dt} = g(x,y), \end{cases} \tag{13}$$

if H is constant on trajectories of (13).

To find a conserved quantity for the system in (12), we solve the differential equation

$$\frac{dy}{dx} = \frac{-x}{y}$$

by separating variables to get

$$\int y\ dy = -\int x\ dx,$$

or

$$\frac{y^2}{2} = -\frac{x^2}{2} + c_2,$$

for some constant of integration c_2, or

$$y^2 = -x^2 + c,$$

where we have written c for $2c_1$; so that

$$x^2 + y^2 = c. \tag{14}$$

Thus, the function $H\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$H(x,y) = x^2 + y^2, \quad \text{for } (x,y) \in \mathbb{R}^2,$$

is a conserved quantity for the system in (12).

The level sets of H are given in (14) for $c \geqslant 0$. These are the origin and concentric circles of radius $\sqrt{c}$ centered at the origin. Some of these are shown in Figure 4. The trajectories of the system in (12) therefore lie on concentric circles about the origin. The direction of the trajectories shown in Figure 4 was obtained by looking at the nullclines. $\square$

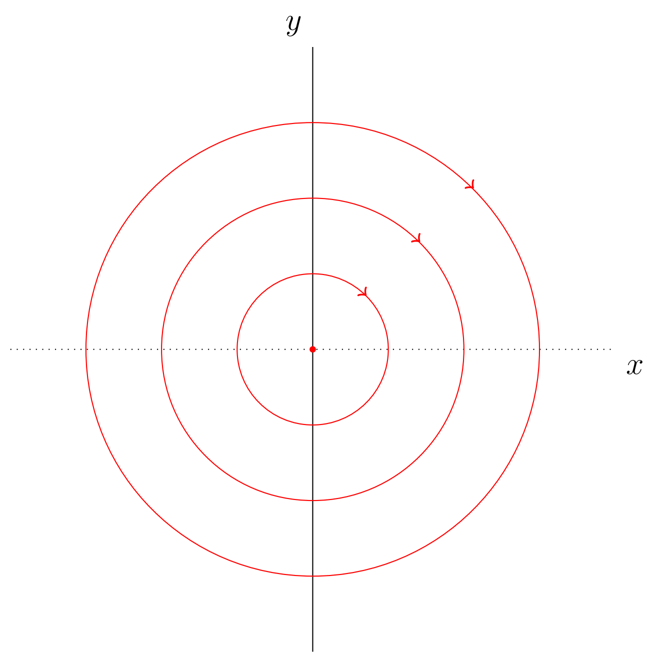

Figure 4: Sketch of Phase Portrait of System (12)