Exam 3 (Part II)

Due on Friday, May 1, 2015

Name: ______________________________

This is the out–of–class portion of Exam 3. There are two questions in this portion of the exam. This is a closed–book and closed–notes exam. Show all significant work and give reasons for all your answers.

Students are expected to work individually on these problems. You may not consult with anyone.

Please, write your name on this page and staple it to your solutions. Turn in your solutions at the start of class on **Friday, May 1, 2015.**

I have read and agree to these instructions. Signature: ______________________

1. The following system of first order differential equations can be interpreted as describing the interaction of two species with population densities x and y:

$$\begin{cases} \dfrac{dx}{dt} = rx\left(1 - \dfrac{x}{L}\right) - \beta xy; \\[2ex] \dfrac{dy}{dt} = \delta xy - \gamma y, \end{cases} \tag{1}$$

for positive parameters r, L, β, γ and δ.

(a) Give the units of each of the parameters r, L, β, γ and δ in (1).

(b) What do the equations in (1) predict about the population density of each species if the other were not present? What effect do the species have on each other? Describe the kind of interaction that the system (1) models.

(c) Introduce dimensionless variables

$$u = \frac{x}{L}, \quad v = \frac{y}{\mu} \quad \text{and} \quad \tau = \frac{t}{\lambda} \tag{2}$$

to write the system in (1) in the dimensionless form

$$\begin{cases} \dfrac{du}{d\tau} = u(1-u) - uv; \\[2ex] \dfrac{dv}{d\tau} = \rho v(u - \alpha), \end{cases} \tag{3}$$

where α and ρ are dimensionless parameters.

(d) Express the scaling parameters μ and λ in (2) in terms of the parameters r and β.

(e) Express the dimensionless parameters α and ρ in (3) in terms of the parameters r, L, δ and γ, and verify that they are dimensionless.

(f) For each of the cases

(i) $0 < \alpha < 1$;

(ii) $\alpha = 1$;

(iii) $\alpha > 1$,

sketch the nullclines of the system (3) in the uv–plane, compute the equilibrium points of the system in the first quadrant, determine the nature of the stability of each equilibrium point, and sketch some possible trajectories.

(g) For each of the cases (i), (ii) and (iii) in part (f) of this problem, describe the different possible long–run behaviors of x and y as $t \to \infty$, and interpret the result in terms of the populations of the two species, and in terms of the original parameters r, L, δ and γ.

2. Consider the population model

$$\frac{dN}{dt} = 0.3N\left(1 - \frac{N}{200}\right)\left(\frac{N}{50} - 1\right), \tag{4}$$

where $N(t)$ denotes the population density at time t.

(a) Write the equation in (4) in the form

$$\frac{dN}{dt} = f(N).$$

Sketch the graph of f versus N for positive values of N, find the equilibrium points of the equation in (4) and use Principle of Linearized Stability (when applicable) to determine the nature of the stability of the equilibrium points.

(b) Sketch the shape of possible solution curves of the equation (4) in the tN–plane for various initial values.

(c) Describe all possible long–term behaviors of solutions of (4). Give interpretations for what the model in (4) predicts.