Solutions to Exam 3 (Part II)

1. The following system of first order differential equations can be interpreted as describing the interaction of two species with population densities x and y:

$$\begin{cases} \dfrac{dx}{dt} = rx\left(1 - \dfrac{x}{L}\right) - \beta xy; \\ \\ \dfrac{dy}{dt} = \delta xy - \gamma y, \end{cases} \tag{1}$$

for positive parameters r, L, β, γ and δ.

(a) Give the units of each of the parameters r, L, β, γ and δ in (1).

Answers:
r has units of 1/time;
L has units of population;
β has units of 1/(population × time);
γ has units of 1/time; and
δ has units of 1/(population × time). □

(b) What do the equations in (1) predict about the population density of each species if the other were not present? What effect do the species have on each other? Describe the kind of interaction that the system (1) models.

Answers: If $y = 0$, the x–species experiences logistic growth with intrinsic growth rate r, and carrying capacity L.
If $x = 0$, then the y–species experiences exponential decay.
In the presence of the y–species, the *per–capita* growth rate of the x–species,

$$\frac{1}{x}\frac{dx}{dt} = r - \frac{rx}{L} - \beta y,$$

decreases with increasing y.
On the other hand, in the presence of the x–species, the *per–capita* growth rate of the y–species,

$$\frac{1}{y}\frac{dy}{dt} = \delta x - \gamma,$$

increases with increasing x.
Hence, the system in (1) models a predator–prey interaction with the population of density x being the prey population, and the population of density y being the predators. □

(c) Introduce dimensionless variables

$$u = \frac{x}{L}, \quad v = \frac{y}{\mu} \quad \text{and} \quad \tau = \frac{t}{\lambda} \tag{2}$$

to write the system in (1) in the dimensionless form

$$\left\{ \begin{aligned} \frac{du}{d\tau} &= u(1-u) - uv; \\ \\ \frac{dv}{d\tau} &= \rho v(u - \alpha), \end{aligned} \right. \tag{3}$$

where α and ρ are dimensionless parameters.

***Solution**:* Differentiate the expression for u in (2) with respect to τ, using the Chain Rule, to get

$$\begin{aligned} \frac{du}{d\tau} &= \frac{du}{dt} \cdot \frac{dt}{d\tau} \\ \\ &= \frac{\lambda}{L}\frac{dx}{dt}, \end{aligned}$$

where we have also used the last expression in (2). Thus, using the first equation in (1),

$$\begin{aligned} \frac{du}{d\tau} &= \frac{\lambda}{L} \cdot \left[rx\left(1 - \frac{x}{L}\right) - \beta xy \right] \\ \\ &= \lambda r u(1-u) - \lambda\mu\beta uv, \end{aligned}$$

where we have also used the definitions of u and v in (2). Therefore, setting

$$\lambda r = 1 \quad \text{and} \quad \lambda\mu\beta = 1, \tag{4}$$

We obtain that

$$\frac{du}{d\tau} = u(1-u) - uv,$$

which is the first equation in (3).

We proceed in a similar way to the previous calculations, this time starting with the second equation in (2), to get

$$\begin{aligned} \frac{dv}{d\tau} &= \frac{dv}{dt} \cdot \frac{dt}{d\tau} \\ \\ &= \frac{\lambda}{\mu}\frac{dy}{dt} \\ \\ &= \frac{\lambda}{\mu}[\delta xy - \gamma y]; \end{aligned}$$

so that, using the definitions of u and v in (2),

$$\frac{dv}{d\tau} = \lambda\delta Luv - \lambda\gamma v,$$

which we can rewrite as

$$\frac{dv}{d\tau} = \lambda\delta Lv\left(u - \frac{\gamma}{\delta L}\right). \tag{5}$$

Setting

$$\rho = \lambda\delta L \quad \text{and} \quad \alpha = \frac{\gamma}{\delta L}, \tag{6}$$

in (5) leads to the second equation in (3). □

(d) Express the scaling parameters μ and λ in (2) in terms of the parameters r and β.

***Solution**:* It follows from (5) that

$$\lambda = \frac{1}{r}, \tag{7}$$

and

$$\mu = \frac{1}{\lambda\beta};$$

so that

$$\mu = \frac{r}{\beta}. \tag{8}$$

It follows from the answer to part (a) and (7) that λ has units of time; similarly, we obtain from (8) that μ has units of population density. □

(e) Express the dimensionless parameters α and ρ in (3) in terms of the parameters r, L, δ and γ, and verify that they are dimensionless.

***Solution**:* The second equation in (7) gives

$$\alpha = \frac{\gamma}{\delta L}. \tag{9}$$

Since γ has units of 1/time, δ has units of 1/time $\times$ pop, and L has units of population, we obtain from (9) that α is dimensionless.

Next, use the first equation in (6) and (7) to get that

$$\rho = \frac{\delta L}{r}, \tag{10}$$

which is dimensionless since both r and δL have units of 1/time. □

(f) For each of the cases

(i) $0 < \alpha < 1$;

(ii) $\alpha = 1$;

(iii) $\alpha > 1$,

sketch the nullclines of the system (3) in the uv–plane, compute the equilibrium points in the first quadrant, determine the nature of the stability of each equilibrium point, and sketch some possible trajectories.

***Solution**:*

(i) Assume that $0 < \alpha < 1$ in (3).

The $\dot{u} = 0$–nullcline for the system (2) are the lines

$$u = 0 \quad \text{(the } v\text{–axis)} \qquad \text{and} \qquad u + v = 1;$$

and the $\dot{v} = 0$–nullcline are the lines

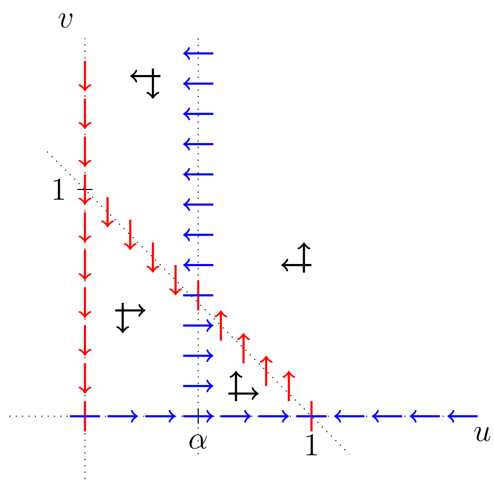


Figure 1: Nullclines of System (3) for the case $0 < \alpha < 1$

$$v = 0 \quad \text{(the } u\text{–axis)} \qquad \text{and} \qquad u = \alpha.$$

These are sketched in Figure 2. We also see in the figure that there are three equilibrium points in the first quadrant:

$$(0,0), \quad (1,0), \qquad \text{and} \qquad (\alpha, 1-\alpha)\,. \tag{11}$$

Next, we compute the linearization of the field

$$F(u,v) = \begin{pmatrix} u - u^2 - uv \\ \rho uv - \alpha\rho v \end{pmatrix}, \qquad \text{for } (u,v) \in \mathbb{R}^2, \tag{12}$$

at the equilibrium points, (u, v), given in (11); namely,

$$DF(\overline{u}, \overline{v}) = \begin{pmatrix} 1 - 2\overline{u} - \overline{v} & -\overline{u} \\ \rho\overline{v} & \rho\overline{u} - \alpha\rho \end{pmatrix}. \tag{13}$$

At the equilibrium point $(0, 0)$ obtain from (13) that

$$DF(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha\rho \end{pmatrix},$$

which has eigenvalues $\lambda_1 = -\alpha\rho < 0$ and $\lambda_2 = 1 > 0$; so that, by the Principle of Linearized Stability, the origin is a saddle point for the system in (3) with $0 < \alpha < 1$.

At the equilibrium point $(1, 0)$, we obtain from (13) that

$$DF(1, 0) = \begin{pmatrix} -1 & -1 \\ 0 & \rho(1 - \alpha) \end{pmatrix},$$

which has eigenvalues $\lambda_1 = -1 < 0$ and $\lambda_2 = \rho(1 - \alpha) > 0$, for $0 < \alpha < 1$; so that, $(1, 0)$ is a saddle point of the system (3) with $0 < \alpha < 1$.

At the equilibrium point $(\alpha, 1 - \alpha)$, we obtain from (13),

$$DF(\alpha, 1 - \alpha) = \begin{pmatrix} -\alpha & -\alpha \\ \rho(1 - \alpha) & 0 \end{pmatrix}. \tag{14}$$

The characteristic polynomial of the matrix in (14) is

$$p(\lambda) = \lambda^2 + \alpha\lambda + \rho\alpha(1 - \alpha);$$

thus, the eigenvalues of the matrix in (14) are given by the expression

$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\rho\alpha(1 - \alpha)}}{2}. \tag{15}$$

We consider two cases:

- $0 < \alpha < 1$ and $\alpha^2 - 4\rho\alpha(1 - \alpha) < 0$; this corresponds to the case

$$\rho > \frac{\alpha}{4(1 - \alpha)}; \tag{16}$$

  and

- $0 < \alpha < 1$ and $\alpha^2 - 4\rho\alpha(1-\alpha) \geqslant 0$, which correspond to the case

$$0 < \rho \leqslant \frac{\alpha}{4(1-\alpha)}. \tag{17}$$

If (16) holds true, (15) yields two complex eigenvalues with nonzero imaginary part, and real part $-\dfrac{\alpha}{2} < 0$; so that, by the Principle of Linearized Stability, the equilibrium point $(\alpha, 1-\alpha)$ is a spiral sink. On the other hand, if (17) holds true, then (15) yields either a single, negative eigenvalue, $-\dfrac{\alpha}{2}$, in the case

$$\rho = \frac{\alpha}{4(1-\alpha)},$$

or two distinct, negative eigenvalues in the case

$$0 < \rho < \frac{\alpha}{4(1-\alpha)}.$$

In all of these cases, $(\alpha, 1-\alpha)$ is a sink for the system in (3) with $0 < \alpha < 1$.
Hence, if $0 < \alpha < 1$ in (1), then $(\alpha, 1-\alpha)$ is an asymptotically stable equilibrium point for the system (1). If

$$\rho > \frac{\alpha}{4(1-\alpha)},$$

$(\alpha, 1-\alpha)$ is a spiral sink; and, if

$$\alpha < \rho \leqslant \frac{\alpha}{4(1-\alpha)},$$

$(\alpha, 1-\alpha)$ is a sink. Figure 2 shows a sketch of the phase portrait of the system in (3) for the case in which $(\alpha, 1-\alpha)$ is a spiral sink. Figure 3 shows a sketch of the phase portrait of the system in (3) for the case in which $(\alpha, 1-\alpha)$ is a sink.

(ii) Assume that $\alpha = 1$ in (3). Then the system (3) becomes

$$\begin{cases} \dfrac{du}{d\tau} &= u(1-u) - uv; \\[2ex] \dfrac{dv}{d\tau} &= \rho v(u-1). \end{cases} \tag{18}$$

The $\dot{u} = 0$–nullcline for the system (18) are the lines

$$u = 0 \quad (\text{the } v\text{–axis}) \quad \text{and} \quad u + v = 1;$$

and the $\dot{v} = 0$–nullcline are the lines

$$v = 0 \quad (\text{the } u\text{–axis}) \quad \text{and} \quad u = 1.$$

These are sketched in Figure 4. We see from the figure that the system in (18) has equilibrium points

$$(0, 0) \quad \text{and} \quad (1, 0) \tag{19}$$

in the first quadrant.

Using the linearization in (13), with $\alpha = 1$, we obtain that the linearization of the system in (18) at the first equilibrium point in (19) has matrix

$$DF(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & -\rho \end{pmatrix},$$

which has eigenvalues $\lambda_1 = -\rho < 0$ and $\lambda_2 = 1 > 0$; so that, by the Principle of Linearized Stability, the origin is a saddle point for the system in (18).

Evaluation of the matrix of the linearization at the second equilibrium point in (19) we obtain, using (13) with $\alpha = 1$,

$$DF(1, 0) = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix},$$

which is not invertible and, therefore, has $\lambda = 0$ as an eigenvalue; consequently, the Principle of Linearized Stability does not apply in this case. We can, however, use the directions of the field in (12), with $\alpha = 1$, outlined in Figure 5.

(iii) Assume that $\alpha > 1$ in (3).

The $\dot{u} = 0$–nullcline for the system (2) with $\alpha > 1$ are the lines

$$u = 0 \quad (\text{the } v\text{–axis}) \quad \text{and} \quad u + v = 1;$$

and the $\dot{v} = 0$–nullcline are the lines

$$v = 0 \quad (\text{the } u\text{–axis}) \quad \text{and} \quad u = \alpha.$$

These are sketched in Figure 6. We also see in the figure that there are two equilibrium points in the first quadrant:

$$(0, 0) \quad \text{and} \quad (1, 0). \tag{20}$$

Using (13) we get that the matrix of the linearization of the system in (3) at the origin is

$$DF(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha\rho \end{pmatrix},$$

which has eigenvalues $\lambda_1 = -\alpha\rho < 0$ and $\lambda_2 = 1 > 0$, which shows that $(0,0)$ is a saddle point, by the Principle of Linearized Stability. Similarly, using (13) we compute the matrix of the linearization of the system in (3) at the second equilibrium point in (20) to get

$$DF(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & -\rho(\alpha-1) \end{pmatrix},$$

which has eigenvalues $\lambda_1 = -1 < 0$ and $\lambda_2 = -\rho(\alpha-1) < 0$, since $\alpha > 1$. Thus, by the Principle of Linearized Stability, $(1,0)$ is a sink. A sketch of the phase–portrait in this case is shown in Figure 7.

□

(g) For each of the cases (i), (ii) and (iii) in part (f) of this problem, describe the different possible long–run behaviors of x and y as $t \to \infty$, and interpret the result in terms of the populations of the two species, and in terms of the original parameters r, L, δ and γ.

Solution:

(i) In view of the expression in (9), the case $\alpha < 1$ corresponds to

$$\frac{\gamma}{\delta L} < 1,$$

or

$$\gamma < \delta L.$$

In this case, according to the sketches in Figure 2 and in Figure 3, any trajectories that starts in the positive portion on the first quadrant in the uv–plane will tend to the equilibrium point

$$(\alpha, 1-\alpha) = \left(\frac{\gamma}{\delta L}, 1 - \frac{\gamma}{\delta L}\right)$$

where we have used (9) again; so that, by virtue of the first two expressions in (2) and (8), if $(x(t), y(t))$ is any trajectory that starts in the positive portion of the first quadrant in the xy–plane will tend to

$$\left(\frac{\gamma}{\delta}, \frac{r}{\beta} - \frac{\gamma r}{\delta\beta L}\right)$$

as $t \to \infty$. Thus, the predator and prey population will coexist limiting values

$$\lim_{t\to\infty} x(t) = \frac{\gamma}{\delta} \quad \text{and} \quad \lim_{t\to\infty} y(t) = \frac{r}{\beta} - \frac{\gamma r}{\delta\beta L}.$$

(ii) The case $\alpha = 1$ corresponds to

$$\gamma = \delta L.$$

In this case, the sketch in Figure 5 shows that, for any initial condition starting in the positive portion of the first quadrant in the uv–plane, all trajectories will tend to $(1, 0)$; so that by virtue of the definitions of u and v in (2), the density of the predator population will tend to 0 as $t \to \infty$, and the density of the prey population will tend towards its carrying capacity, L, as $t \to \infty$.

(iii) The case $\alpha > 1$ corresponds to

$$\gamma > \delta L.$$

In this case, the sketch in Figure 7 shows that, for any initial condition starting in the positive portion of the first quadrant in the uv–plane, all trajectories will tend to $(1, 0)$; so that by virtue of the definitions of u and v in (2), the density of the predator population will tend to 0 as $t \to \infty$, and the density of the prey population will tend towards its carrying capacity, L, as $t \to \infty$, in this case as well.

□

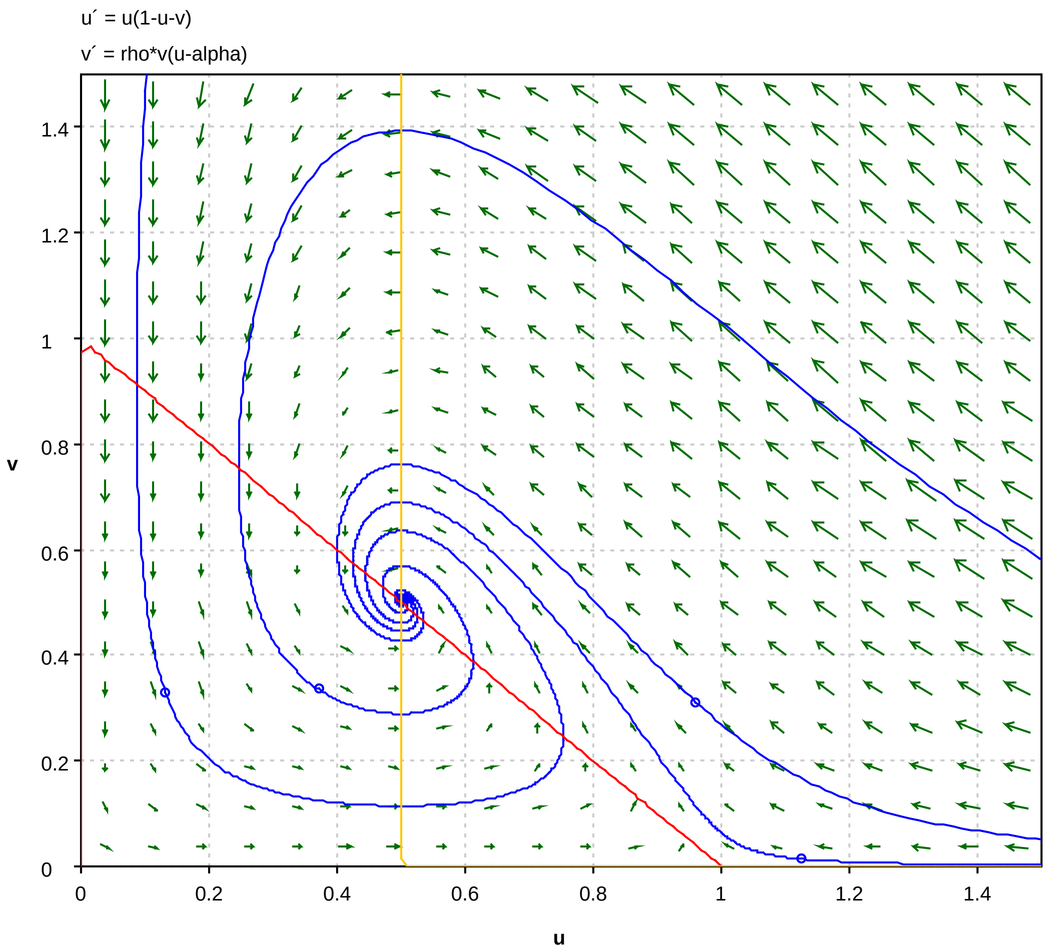


Figure 2: Sketch of Phase Portrait for System (3) with $\alpha = 0.5$ and $\rho = 2$

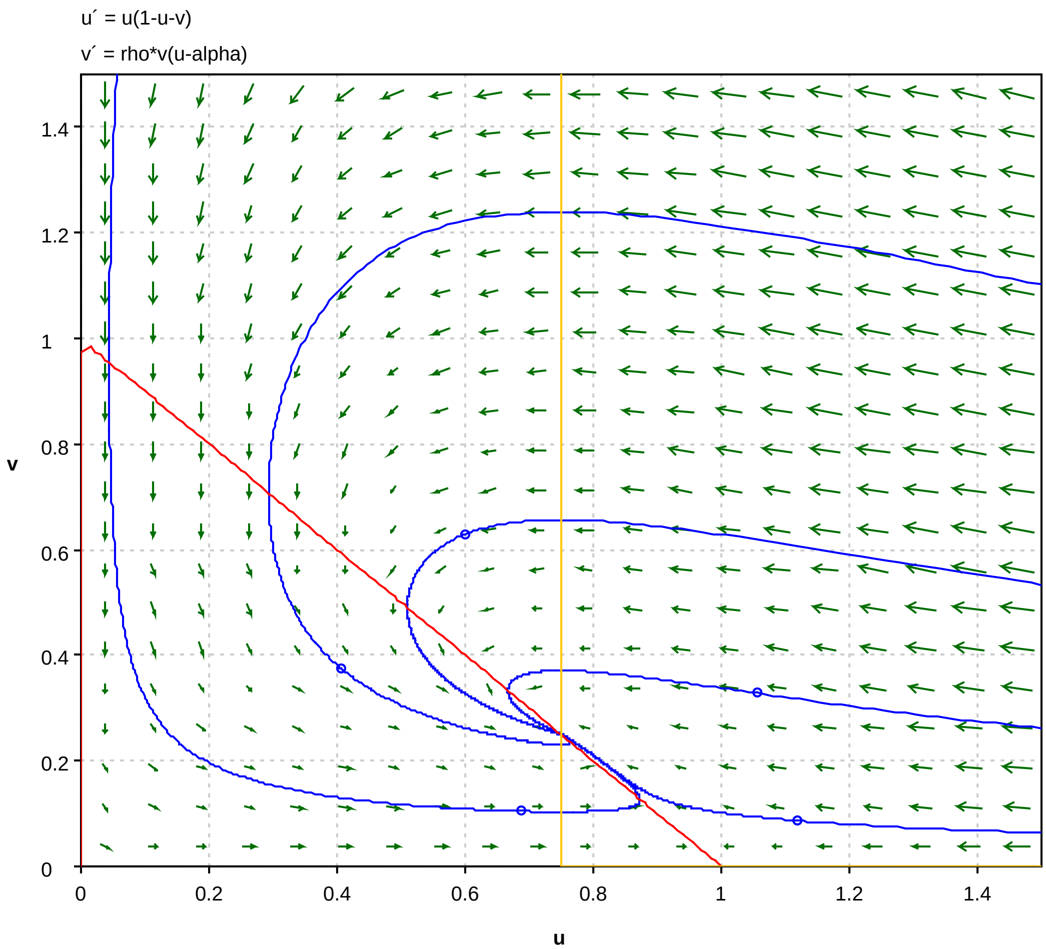


Figure 3: Sketch of Phase Portrait for System (3) with $\alpha = 0.75$ and $\rho = 0.7$

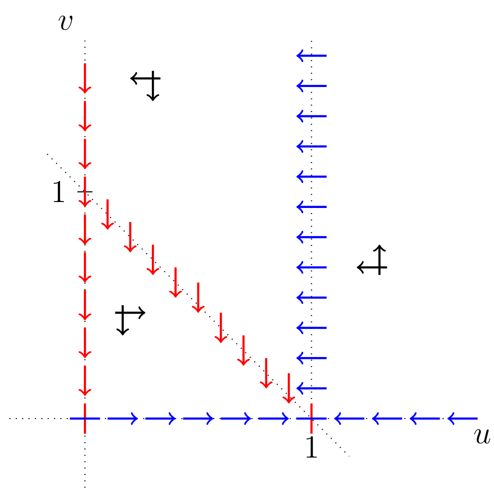


Figure 4: Nullclines of System (18)

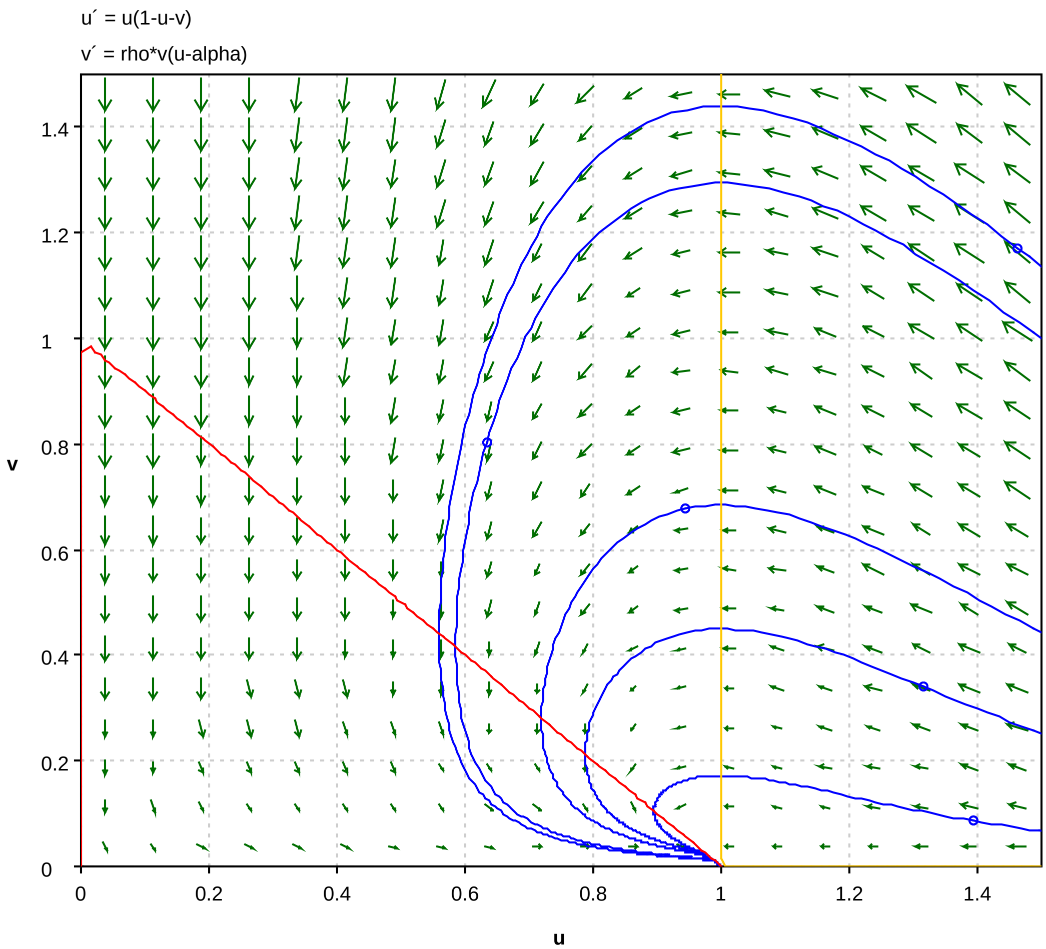


Figure 5: Sketch of Phase Portrait for System (18)

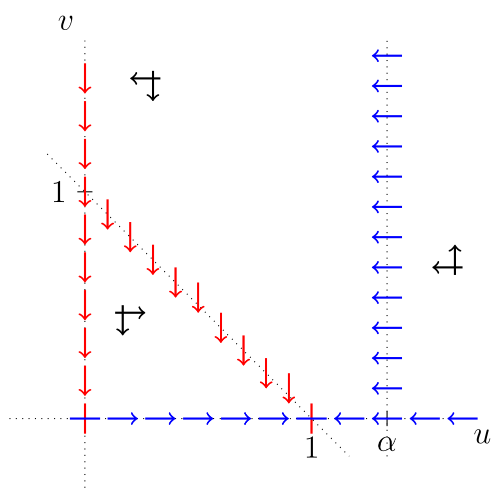


Figure 6: Nullclines of System (3) with $\alpha > 1$

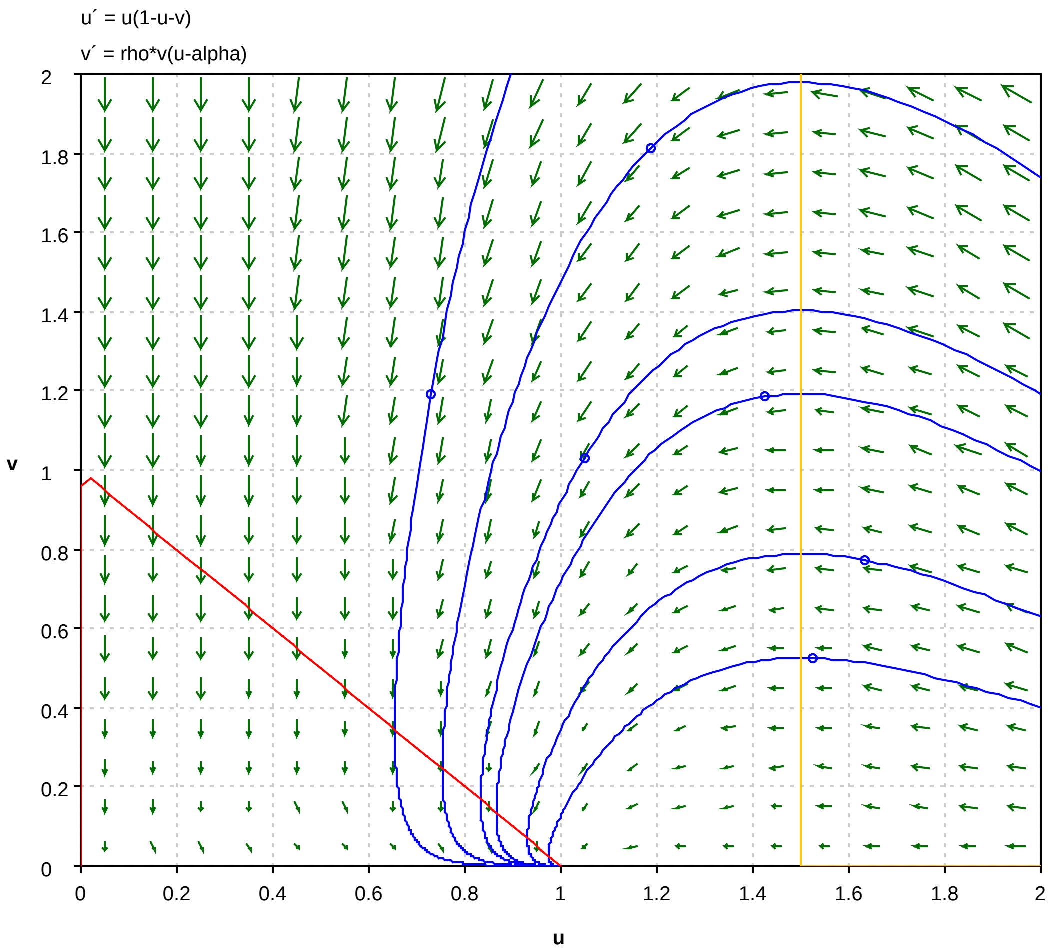


Figure 7: Sketch of Phase Portrait for System (3) with $\alpha = 1.5$ and $\rho = 5$

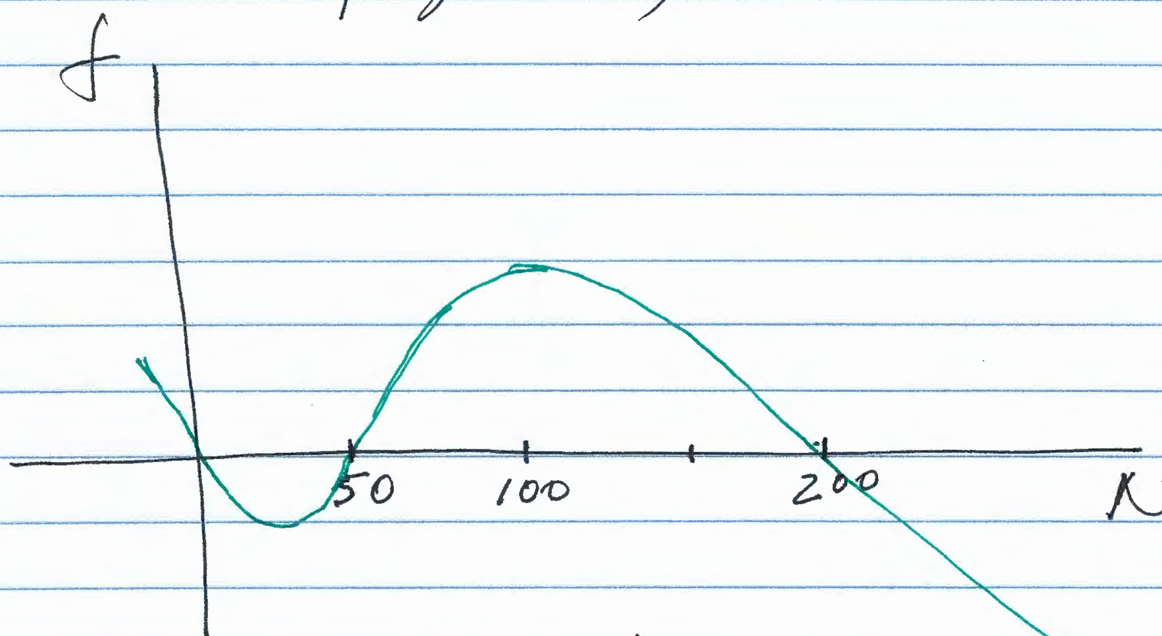
2(a) Let $f(N) = 0.3N\left(1 - \frac{N}{200}\right)\left(\frac{N}{50} - 1\right)$, which we can rewrite as

$$f(N) = -3\times10^{-5}\, N(N-200)(N-50)$$

Thus, the equation $\frac{dN}{dt} = f(N)$ has equilibrium points at

$\bar{N}_1 = 0$, $\bar{N}_2 = 50$ and $\bar{N}_3 = 200$

A sketch of the graph of f vs. N is shown below (note that $f(N)$ is a cubic polynomial)

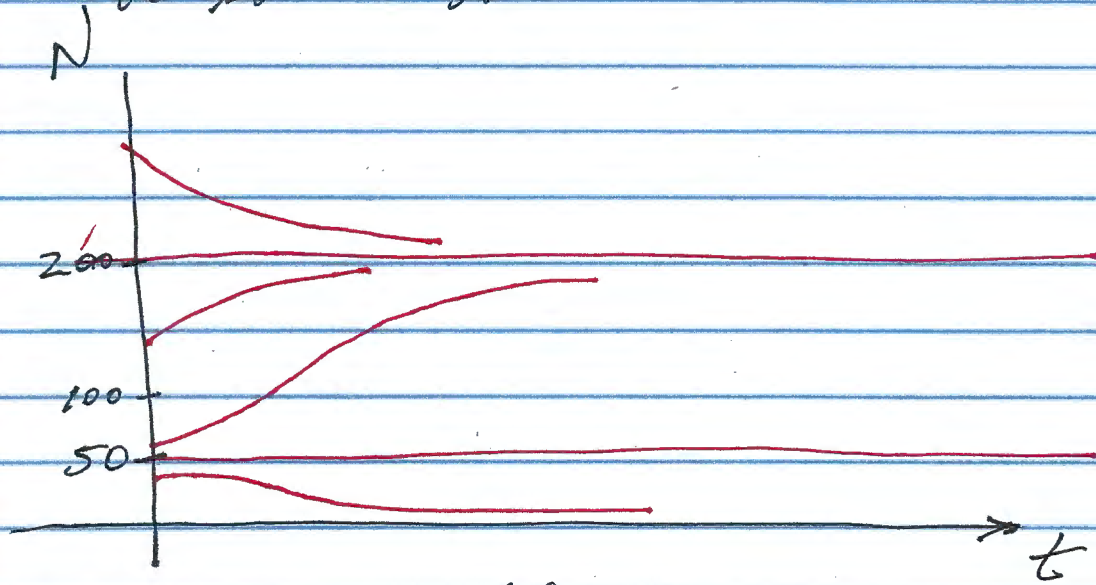


We see from the sketch that $f(\bar{N}_1) < 0$; so that, by the Principle of linearized stability, $\bar{N}_1 = 0$ is asymptotically stable.

Similarly, since $f'(\bar{N}_2) > 0$, $\bar{N}_2 = 50$

is unstable by the Principle of Linearized Stability; and, since $f'(\bar{N}_3) < 0$, $\bar{N}_2 = 200$ is asymptotically stable, by the Principle of Linearized Stability.

(b) A sketch of possible solutions is shown below



(c) The model in Equation (4) in Part II of Exam 3 predicts that if $0 < N(0) < 50$, then $\lim_{t\to\infty} N(t) = 0$; so that the population will eventually die out. On the other hand, if $N(0) > 50$,

$$\lim_{t\to\infty} N(t) = 200.$$