Review Problems for Exam 3

1. Find the equilibrium solutions of following autonomous differential equations and determine the nature of the satiability of the equilibrium solutions. Sketch some possible solution curves. If possible, describe the long–term behavior of the solutions.

   (a) $\dfrac{dx}{dt} = (x-3)(x-5)$

   (b) $\dfrac{dx}{dt} = (1-x)(x-2)^2$

2. The following equation models the evolution of a population that is being harvested at a constant rate:

$$\frac{dN}{dt} = 2N - 0.01N^2 - 75.$$

   Find equilibrium solutions and sketch a few possible solution curves. According to model, what will happen if at time $t = 0$ the initial population densities are 40, 60, 150, or 170.

3. For the following systems, sketch nullclines, find equilibrium points, determine their stability properties, and describe the local behavior trajectories near the equilibrium points. Sketch the phase portraits.

   (a) $\begin{cases} \dot{x} &= x^2 - y^2 - 1; \\ \dot{y} &= 2y, \end{cases}$

   (b) $\begin{cases} \dot{x} &= y - y^2 + 2; \\ \dot{y} &= 2x^2 - 2xy, \end{cases}$

   (c) $\begin{cases} \dot{x} &= 4 - 2y; \\ \dot{y} &= 12 - 3x^2. \end{cases}$

4. Assume that the system $\begin{cases} \dot{x} &= f(x,y); \\ \dot{y} &= g(x,y), \end{cases}$ where f and g are real valued functions of two variables with continuous partial derivatives, is a gradient system.

(a) Verify that $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$.

(b) Use your result from part (a) to show that the system $\left\{\begin{array}{rcl}\dot{x} & = & x^2 + 3xy;\\ \dot{y} & = & 2x + y^3,\end{array}\right.$ is not a gradient system.

5. **Negative Gradient Flows.** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ denote a twice differentiable function with continuous partial derivatives. Consider the negative gradient system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = -\nabla f(x, y). \tag{1}$$

(a) Let $(x(t), y(t))$ denote a solution curve of the system in (1) that contains no equilibrium points of (1). Show that f is strictly decreasing (with increasing t) along this trajectory.

(b) Let $(x(t), y(t))$ denote a solution curve of the system in (1) that contains no equilibrium points of (1). Explain why this trajectory cannot be a cycle.

6. **The Linear Pendulum Equation.** The pendulum equation (without friction),

$$\ell\ddot{\theta} = -g\sin(\theta), \tag{2}$$

can be linearized about the equilibrium position $\overline{\theta} = 0$ to yield the linear equation

$$\ell\ddot{\theta} = -g\theta. \tag{3}$$

The equation in (3) is the linearization of the equation in (2) and corresponds to oscillations of very small amplitude.

(a) Nondimsionalize the equation in (3) by introducing a dimensionless variable

$$\tau = \frac{t}{\lambda}.$$

What is the value of the parameter λ in terms of ℓ and g?

(b) Solve the equation obtained in part (a) by first performing a phase–plane analysis.

(c) Compute the period, T, of oscillations of solutions of (3) in terms of ℓ and g.

7. **Modeling the Spread of a Disease.** In a simple model for a disease that is spread through infections transmitted between individuals in a population, the population is divided into three compartments pictured in Figure 1. In the first

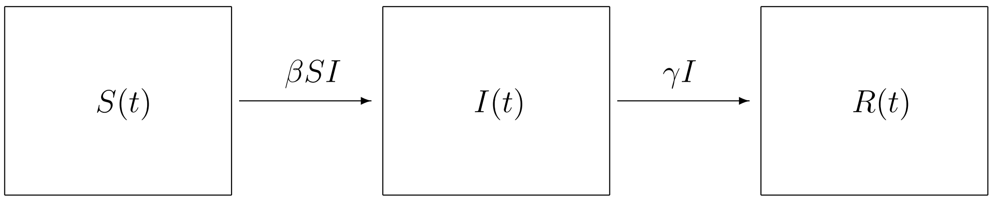

Figure 1: SIR Compartments

compartment, $S(t)$ denotes the number of individuals in a population that are susceptible to acquiring the disease; in the second compartment, $I(t)$ denotes the number of infected individual who can also infect others; and, in the third compartment, $R(t)$ denotes the number of individuals who had the disease and who have recovered from it; they can no longer get infected.

The arrows between compartments indicate the rates at which individuals flow from one compartment to the other. For instance, the arrow between the first two compartments indicates the transmission rate of the disease; it is assumed that the rate at which susceptible individuals get infected is proportional to product of number of susceptible individuals and the number of infected individuals with constant of proportionality $\beta > 0$. The rate at which infected individuals recover is indicated by the arrow between the last two compartments; it is assumed that this rate is proportional to the number of infected individuals, with constant of proportionality $\gamma > 0$.

(a) What are the units for β and γ?

(b) Use conservation principles to derive a system of differential equations for the functions S, I and R, assuming that they are differentiable, of the form

$$\begin{cases} \dfrac{dS}{dt} = f(S, I, R, \beta, \gamma); \\ \dfrac{dI}{dt} = g(S, I, R, \beta, \gamma); \\ \dfrac{dR}{dt} = h(S, I, R, \beta, \gamma), \end{cases} \tag{4}$$

where f, g and h are continuous functions that have continuous partial derivatives with respect to S, I and R. The system in (4) is known in the literature as the Kermack–McKendrick SIR model. It first appeared in the scientific literature in 1927.

(c) Deduce that the system in (4) implies that the total number of individuals in the population,

$$N(t) = S(t) + I(t) + R(t),$$

remains constant. Denote $N(t)$ by N, where N is a constant, for all t.

(d) Explain why the result of part (c) implies that the study of the system (4) reduces to the study of the two–dimensional system

$$\begin{cases} \dfrac{dS}{dt} &= f(S, I, R, \beta, \gamma); \\ \\ \dfrac{dI}{dt} &= g(S, I, R, \beta, \gamma). \end{cases} \tag{5}$$

(e) Introduce dimensionless variables

$$x = \frac{S}{N}, \quad y = \frac{I}{N}, \qquad \text{and} \quad \tau = \frac{t}{\lambda},$$

for some scaling factor, λ, in units of time, in order to write the system (5) in dimensionless form.

(f) Analyze the system obtained in part (e). What does the model in (4) predict about the spread of the disease in terms of the initial conditions $S(0) = S_o$, $I(0) = I_o$, $R(0) = 0$, and the parameters β, γ and N? Under which conditions will the number of infected individuals increase (an epidemic outbreak), or decrease?