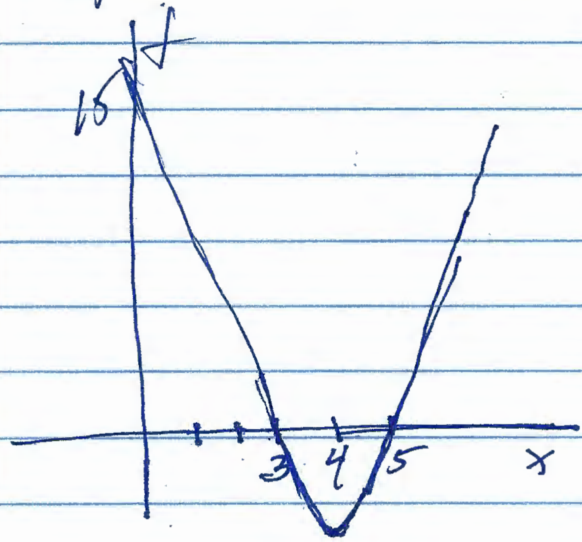
Review Problems for Exam 3

(1) (a) Write $f(x) = (x-3)(x-5)$ and sketch f versus x



Equilibrium points:

$\bar{x}_1 = 3$, $\bar{x}_2 = 5$

$f'(\bar{x}_1) < 0$, so $\bar{x}_1 = 3$ is asymptotically stable

$f'(\bar{x}_2) > 0$, so $\bar{x}_2 = 5$ is unstable

	$x<3$	$3<x<4$	$4<x<5$	$x>5$
$f(x)$:	+	−	−	+
$f'(x)$:	−	−	+	+
$x''(t)$:	−	+	−	+
concavity	$\frown$	$\smile$	$\frown$	$\smile$

$x(t)$ increases on $(-\infty, 3)$ and $(5, +\infty)$

$x(t)$ decreases on $(3, 5)$

Graph of f is concave up on $(3,4)$ and $(5, +\infty)$

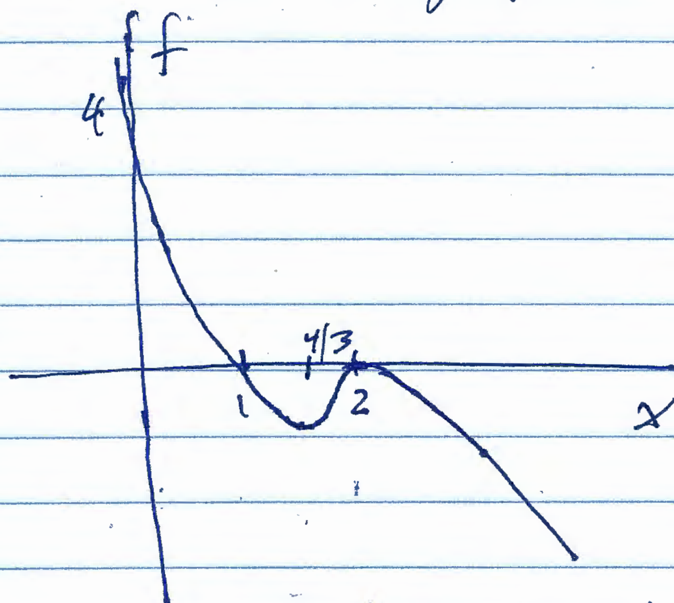
Graph of f is concave down on $(-\infty, 3)$ and $(4,5)$

Graphs of possible solutions

Depending on the initial conditions $x(t) \to 3$ as $t \to \infty$, or $x(t) \to \infty$ as $t \to \infty$.

1(b) $f(x) = (1-x)(x-2)^2$

Sketch the graph of f versus x



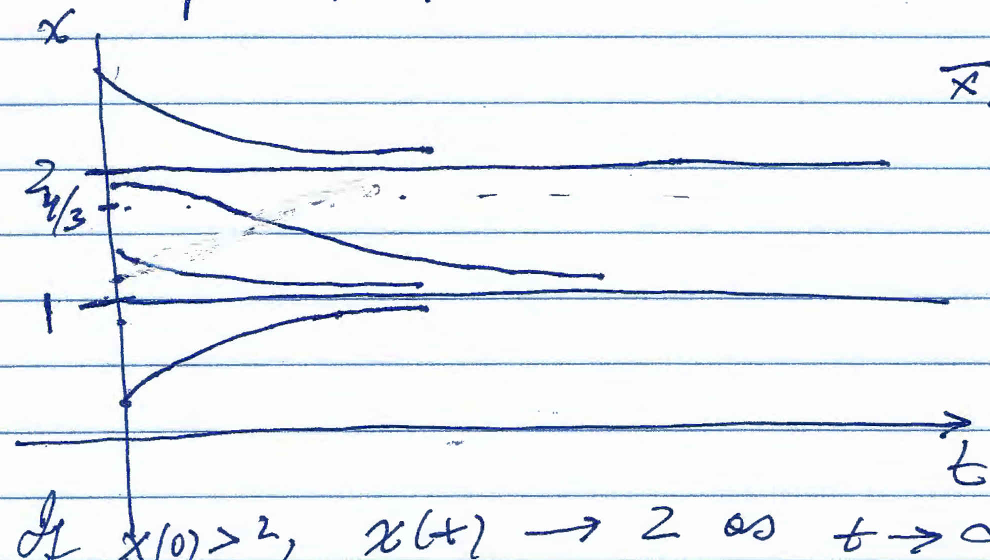
Equilibrium points: $\bar{x}_1 = 1$, $\bar{x}_2 = 2$

$f'(\bar{x}_1) < 0 \Rightarrow \bar{x}_1$ is asymptotically stable by the PLS.

Since $f'(\bar{x}_2) = 0$, the PLS does not apply.

$f(x)$:	$+$	$-$	$-$	$-$
$f'(x)$:	$-$	$-$	$+$	$-$
x'' :	$-$	$+$	$-$	$+$
Concavity	$\cap$ 1	$\cup$ 4/3	$\cap$ 2	$\cup$

Graphs of possible solutions



$\bar{x}_2$ is unstable

If $x(0) > 2$, $x(t) \to 2$ as $t \to \infty$.

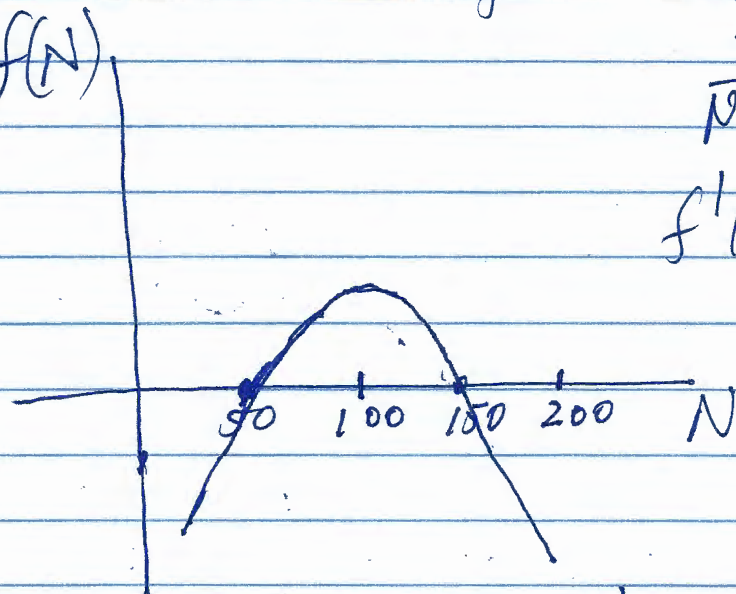
If $x(0) < 2$, $x(t) \to 1$ as $t \to \infty$.

(2) Write $f(N) = 2N - 0.01N^2 - 75$

or $f(N) = -0.01(N^2 - 200N + 7500)$

$= -0.01(N-50)(N-150)$

and sketch f versus N



Equilibrium pts

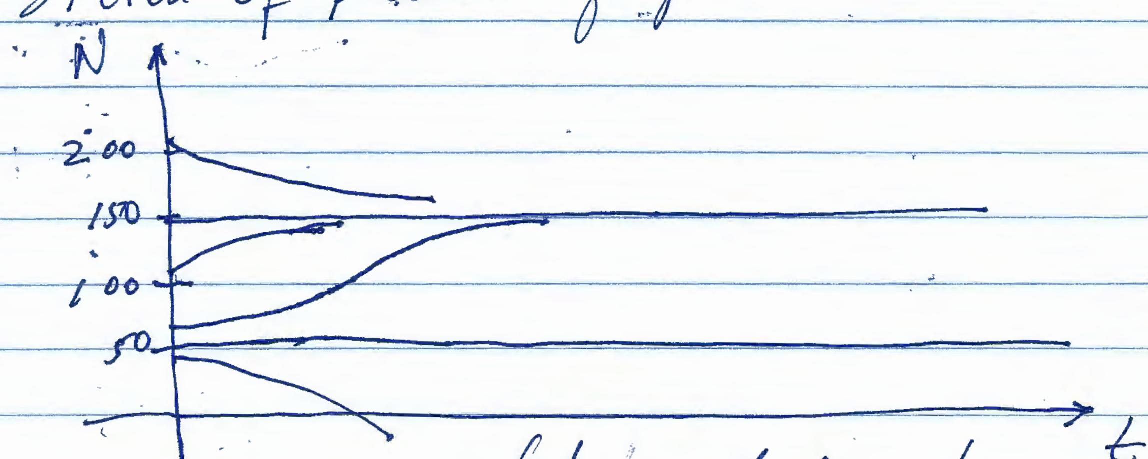
$\bar{N}_1 = 50$, $\bar{N}_2 = 150$

$f'(\bar{N}_1) > 0 \Rightarrow$
$\bar{N}_1$ is unstable

$f'(\bar{N}_2) < 0 \Rightarrow$
$\bar{N}_2$ is asymptotically stable.

	$N<50$	$50<N<100$	$100<N<150$	$N>150$
$f(N)$	$-$	$+$	$+$	$-$
$f'(N)$	$+$	$+$	$-$	$-$
N''	$-$	$+$	$-$	$+$
concavity	$\cap$	$\cup$	$\cap$	$\cup$

Sketch of possible graphs



If $N(0) < 50$, population will die out in finite time

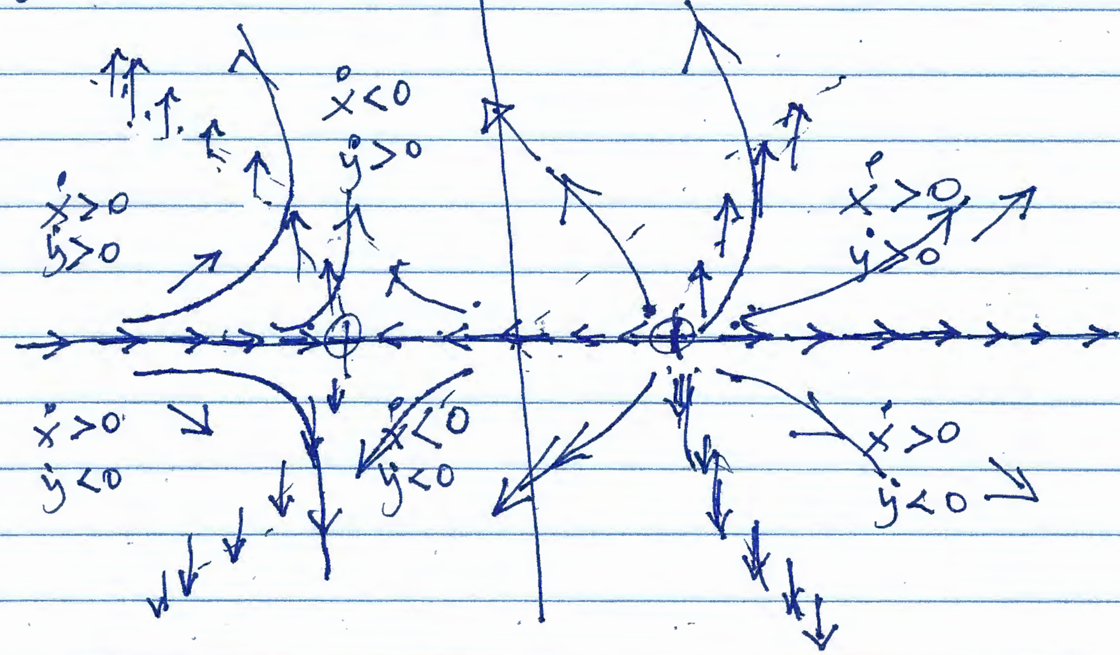
If $N(0) > 50$, $N(t) \to 150$ as $t \to \infty$.

3(a) $\begin{cases} \dot{x} = x^2 - y^2 - 1 \\ \dot{y} = 2y. \end{cases}$

Nullclines:

$\dot{x} = 0$ - nullcline $\quad x^2 - y^2 = +1$ (hyperbola)

$\dot{y} = 0$ - nullcline $\quad y = 0$ (x axis)



Equilibrium points: $(-1, 0)$ & $(1, 0)$

Local Analysis.

$$F(x,y) = \begin{pmatrix} x^2 - y^2 - 1 \\ 2y \end{pmatrix}$$

$$DF(x,y) = \begin{pmatrix} 2x & -2y \\ 0 & 2 \end{pmatrix}$$

$$DF(-1,0) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow (-1,0) \text{ is a saddle point}$$

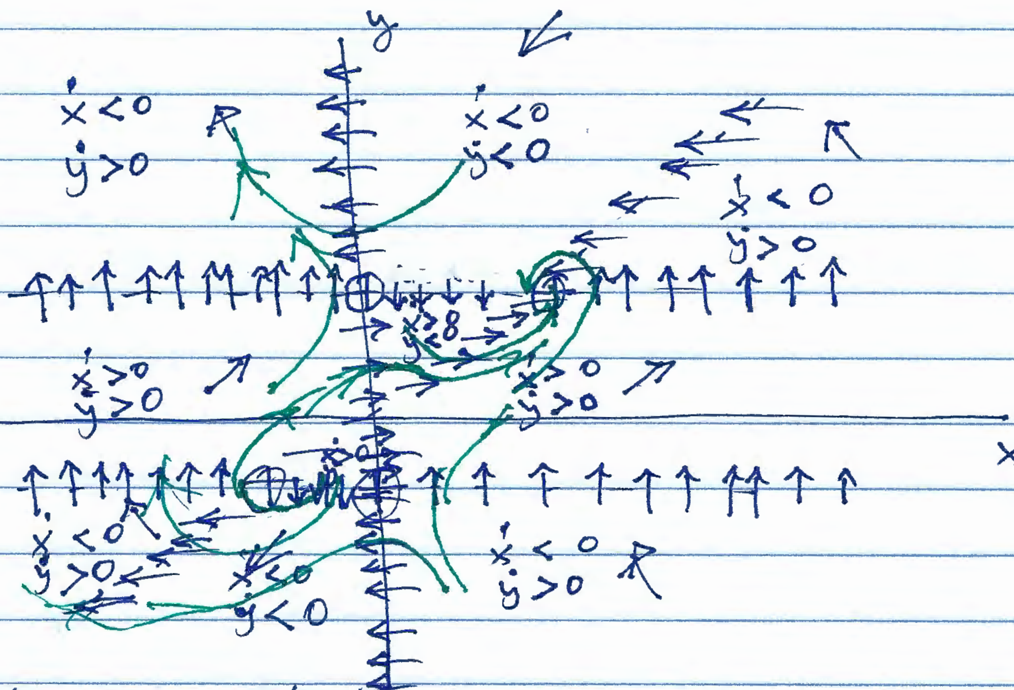
$$DF(1,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow (1,0) \text{ is a source}$$

3(b) $\begin{cases} \dot{x} = y - y^2 + 2 \\ \dot{y} = 2x(x-y) \end{cases}$

Nullclines:

$\dot{x}=0$-nullclines: lines $y=-1$, $y=2$

$\dot{y}=0$-nullclines: $x=0$ (y-axis), $y=x$



Equilibrium points: $(-1,-1)$, $(0,-1)$, $(0,2)$ & $(2,2)$

Local analysis:

$$F(x,y) = \begin{pmatrix} y - y^2 + 2 \\ 2x^2 - 2xy \end{pmatrix}$$

$$DF(x,y) = \begin{pmatrix} 0 & 1-2y \\ 4x-2y & -2x \end{pmatrix}$$

$$DF(-1,-1) = \begin{pmatrix} 0 & 3 \\ -2 & 2 \end{pmatrix}$$

Eigenvalues: $\lambda = 1 \pm i\sqrt{5}$

Thus, $(-1,-1)$ is a spiral ~~sink~~ source

$$DF(0,-1) = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$$

Eigenvalues: $\lambda = \pm\sqrt{6}$

Thus, $(0,-1)$ is a saddle point

$$DF(0,2) = \begin{pmatrix} 0 & -3 \\ -4 & 0 \end{pmatrix}$$

Eigenvalues: $\lambda = \pm\sqrt{12}$

Thus, $(0,2)$ is a saddle point

$$DF(2,2) = \begin{pmatrix} 0 & -3 \\ 4 & -4 \end{pmatrix}$$

Eigenvalues: $\lambda = -2 \pm i2\sqrt{2}$

Thus, $(2,2)$ is a spiral sink

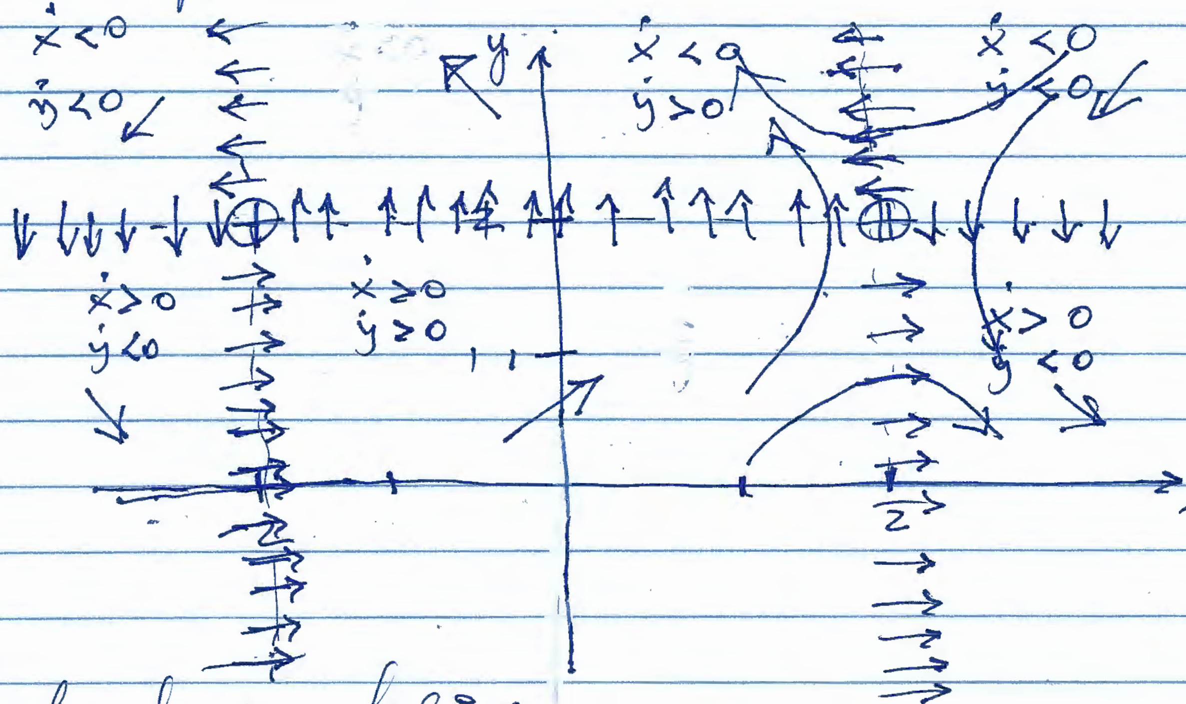
3(c) $\begin{cases} \dot{x} = 4 - 2y \\ \dot{y} = 12 - 3x^2 \end{cases}$

Nullclines:

$\dot{x} = 0$ - nullcline: line $y = 2$

$\dot{y} = 0$ - nullcline: lines $x = -2$ & $x = 2$

Equilibrium points: $(-2, 2)$ & $(2, 2)$



Local analysis:

$$F(x,y) = \begin{pmatrix} 4 - 2y \\ 12 - 3x^2 \end{pmatrix}$$

$$DF(x,y) = \begin{pmatrix} 0 & -2 \\ -6x & 0 \end{pmatrix}$$

$$DF(-2,2) = \begin{pmatrix} 0 & -2 \\ 12 & 0 \end{pmatrix}$$ Eigenvalues: $\lambda = \pm i2\sqrt{6}$
PLS does not apply

$$DF(2,2) = \begin{pmatrix} 0 & -2 \\ -12 & 0 \end{pmatrix}$$ Eigenvalues: $\lambda = \pm 2\sqrt{6}$
so $(2,2)$ is a saddle pont