Assignment #13

Due on Wednesday, March 11, 2015

Read Section 14.4, on *Gradients and Directional Derivatives in the Plane,* in Calculus: Multivariable, by McCallum, Hughes–Hallett, Gleason, et al.

Read Section 14.6, on *The Chain Rule,* in Calculus: Multivariable, by McCallum, Hughes–Hallett, Gleason, et al.

Read Section 13.3, on *The Dot Product,* in Calculus: Multivariable, by McCallum, Hughes–Hallett, Gleason, et al.

Background and Definitions.
The Chain Rule (Version II). Let $f\colon D \to \mathbb{R}$ be a real–valued function defined on some domain, D, in the xy–plane, and let $\overrightarrow{r}\colon I \to \mathbb{R}^2$, for some open interval I, denote a differentiable path with $\overrightarrow{r}(t) \in D$ for all $t \in I$. Suppose that the partial derivatives of f exist and are continuous in D. Then, for any $t \in I$,

$$\frac{d}{dt}[f(\overrightarrow{r}(t))] = \nabla f(\overrightarrow{r}(t)) \cdot \overrightarrow{r}'(t),$$

where ∇f denotes the gradient of f; that is,

$$\nabla f(x,y) = \frac{\partial f}{\partial x}(x,y)\widehat{\imath} + \frac{\partial f}{\partial y}(x,y)\widehat{\jmath}, \quad \text{for all } (x,y) \in \mathbb{R}^2,$$

$\overrightarrow{r}'(t)$ is the derivative of the path $\overrightarrow{r}$, and the dot between ∇f and $\overrightarrow{r}$ indicates the dot product of the two vectors.

Do the following problems

1. Let $f\colon D \to \mathbb{R}$ be a real–valued function defined on some domain, D, in the xy–plane, and let $\overrightarrow{r}\colon I \to \mathbb{R}^2$, for some open interval I, denote a parametrization of a contour curve for the function f; that is,

$$f(\overrightarrow{r}(t)) = c, \quad \text{for all } t \in I.$$

   Apply the Chain Rule to obtain

$$\nabla f(\overrightarrow{r}(t)) \cdot \overrightarrow{r}'(t) = 0, \quad \text{for all } t \in I;$$

   deduce therefore that the gradient of f is perpendicular to the level curves of f.

2. Let $\widehat{u}$ denote a unit vector and put $\overrightarrow{r}(t) = (x_o, y_o) + t\widehat{u}$ for all $t \in \mathbb{R}$. Let $f\colon D \to \mathbb{R}$ be a real–valued function defined on some domain, D, in the xy–plane that contains the point (x_o, y_o).

   Apply the Chain Rule to compute $\dfrac{d}{dt}[f(\overrightarrow{r}(t))]$ at $t = 0$. Explain why this yields the directional derivative of f at (x_o, y_o) in the direction of $\widehat{u}$. We denote this number by $D_{\widehat{u}}f(x_o, y_o)$.

3. Let $\widehat{u}$ denote a unit vector and let $f\colon D \to \mathbb{R}$ be a real–valued function defined on some domain, D, in the xy–plane. Define $D_{\widehat{u}}f$ as in Problem 2; so that

   $$D_{\widehat{u}}f(x, y) = \nabla f(x, y) \cdot \widehat{u}, \quad \text{for all } (x, y) \in D.$$

   Deduce that

   $$D_{\widehat{u}}f(x, y) = \|\nabla f(x, y)\| \cos\theta, \quad \text{for all } (x, y) \in D, \tag{1}$$

   where θ is the angle that $\nabla f(x, y)$ makes with the unit vector $\widehat{u}$.

   Conclude from (1) that the rate of change of f at (x, y) is the largest in the direction of the gradient of f at (x, y).

4. Let $f(x, y) = 3xy + y^2$ for all $(x, y) \in \mathbb{R}^2$. Give The direction of maximum rate of change of f at $(2, 3)$.

5. Let $f(x, y) = 3xy + y^2$ for all $(x, y) \in \mathbb{R}^2$. Give The direction in which f is decreasing the fastest at $(2, 3)$.