Assignment #14

Due on Wednesday, March 25, 2015

Read Section 14.3, on *Local Linearity and the Differential,* in Calculus: Multivariable, by McCallum, Hughes–Hallett, Gleason, et al.

Background and Definitions.

- **Linear approximation of a real valued function of two variables**. Let $f\colon D \to \mathbb{R}$ be a real–valued function defined on some domain, D, in the xy–plane, and let (x_o, y_o) denote a point in D. Suppose that the partial derivatives of f exist and are continuous in D. Then, the linear approximation for f at (x_o, y_o) is the linear function $L\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$L(x,y) = f(x_o,y_o) + \frac{\partial f}{\partial x}(x_o,y_o)\cdot(x-x_o) + \frac{\partial f}{\partial y}(x_o,y_o)\cdot(y-y_o), \quad \text{for } (x,y)\in\mathbb{R}^2.$$

  $L(x,y)$ approximates $f(x,y)$ when (x,y) is very close to (x_o,y_o). We write

$$f(x,y) \approx f(x_o,y_o) + \frac{\partial f}{\partial x}(x_o,y_o)\cdot(x-x_o) + \frac{\partial f}{\partial y}(x_o,y_o)\cdot(y-y_o), \qquad (1)$$

  for (x,y) in D sufficiently close to $(x_o, y_o$.

- **Tangent plane to the graph of a function of two variables**. The graph of the equation

$$z = L(x,y)$$

  is a plane through (x_o,y_o) called the tangent plane to the graph of $z = f(x,y)$ at the point (x_o,y_o).

- **The differential of a function of two variables**. Writing $z = f(x,y)$ and $z_o = f(x_o,y_o)$, we can rewrite (1) as

$$z \approx z_o \frac{\partial f}{\partial x}(x_o,y_o)\cdot(x-x_o) + \frac{\partial f}{\partial y}(x_o,y_o)\cdot(y-y_o),$$

  when (x,y) is sufficiently close to (x_o,y_o), or

$$\Delta z \approx \frac{\partial f}{\partial x}(x_o,y_o)\Delta x + \frac{\partial f}{\partial y}(x_o,y_o)\Delta y, \qquad (2)$$

  where $\Delta z = z - z_o$, $\Delta y = y - y_o$ and $\Delta x = x - x_o$. The expression in (2) motivates the definition of the differential of f:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Do the following problems

1. Give the equation of the tangent plana to the graph of

$$z = \frac{1}{2}x^2 + 2y^2$$

at the point $(2, 1, 4)$.

2. Give the linear approximation to the function given by $f(x, y) = x^2y$, for $(x, y) \in \mathbb{R}^2$, at the point $(3, 1)$.

3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \sqrt{x^2 + y^2}, \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

   (a) Give the differential of f at the point $(3, 4)$.

   (b) Use the differential of f at $(3, 4)$ to estimate $f(2.98, 4.01)$.

4. Assume that the temperature in an unevenly heated plate is given by $T(x, y)$ °C at every point (x, y) in the plate, where T is a function of two variables with continuous partial derivatives T_x and T_y. Assume that $T(2, 1) = 135$ °C, and that the partial derivatives of T at $(2, 1)$ have values $T_x(2, 1) = 16$ and $T_y(2, 1) = -15$. Estimate the temperature at the point $(2.04, 0.97)$.

5. Let $p(A, D)$ denote the expression given the number π, where A denotes the area enclosed by a circle and D the diameter of the circle.

   (a) Give and expression of $p(A, D)$.

   (b) Compute the differential of p.

   (c) Assume that a percent error of 0.001 can be made when measuring the area enclosed by the circle, and a percent error of 0.0005 can be made when measuring the diameter. Use the differential computed in part (b) to estimate the error in computing π.