Solutions to Review Problems for Exam 2

1. Put $f(x, y) = 4 - \sqrt{x^2 + y^2}$.

   (a) Give the domain of f.

   Answer: $\text{Dom}(f) = \mathbb{R}^2$. □

   (b) Sketch a contour plot for the graph of f.

   Answer: The level sets of the function f are the graphs of the equations

   $$4 - \sqrt{x^2 + y^2} = c, \quad \text{for } c \leqslant 4,$$

   or

   $$x^2 + y^2 = (4 - c)^2, \quad \text{for } c \leqslant 4.$$

   These are concentric circles around the origin with radius $4 - c$ for $c \leqslant 4$.
   A sketch of the contour diagram is shown in Figure 1. □

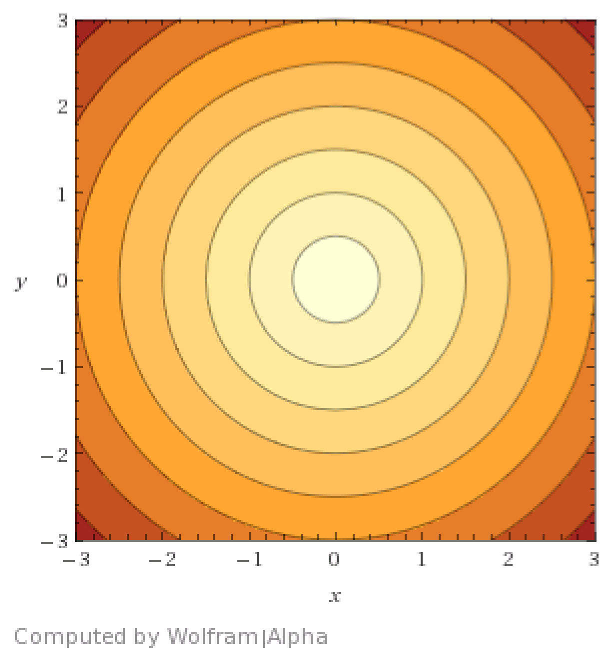

Figure 1: Sketch of Contour Diagram in Problem 1

(c) Sketch the graph of f.

***Solution**:* To sketch the graph of $z = f(x, y)$ it is helpful to look at the sections in the xz–plane and the yz–plane.

In the yz–plane, the section is the graph of

$$z = 4 - \sqrt{y^2},$$

or

$$z = 4 - |y|,$$

which yields a pair of half–lines emanating from the point $(0, 0, 4)$; similarly, in the xz–plane, we get

$$z = 4 - |x|,$$

which is also a pair of half–lines emanating from the point $(0, 0, 4)$.

Since the level sets are circles, we conclude that the graph of f is a circular cone with vertex at $(0.0, 4)$. A sketch of this cone is shown in Figure 2. □

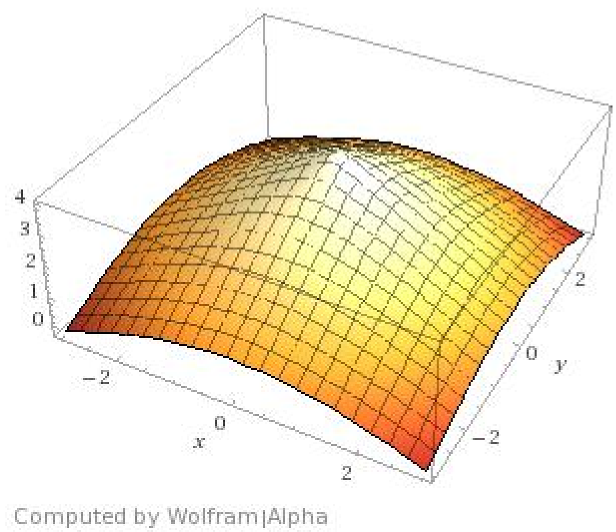

Figure 2: Sketch of graph of f in Problem 1

2. Let $f(x, y) = \sqrt{x^2 + 2y^2}$ for all $(x, y) \in \mathbb{R}^2$. Sketch a contour plot for the function f.

   ***Solution**:* The contour curves are graphs of the equations

$$\sqrt{x^2 + 2y^2} = c, \qquad \text{for } c \geqslant 0,$$

or

$$x^2 + 2y^2 = c^2,$$

which yield ellipses with major axis along the x–axis. Some of these are shown in Figure 3. □

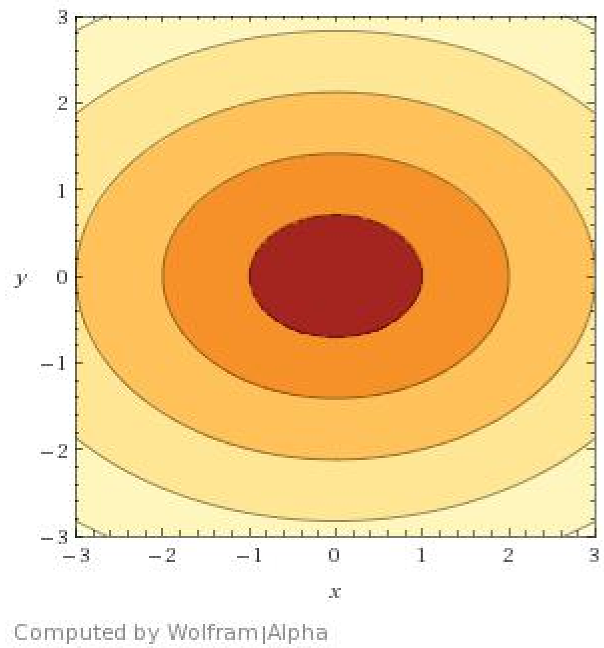

Figure 3: Sketch of Contour Diagram in Problem 2

3. Let $f(x, y) = x^2 - y^2$ for all $(x, y) \in \mathbb{R}^2$. Sketch a contour plot for the function f.

   Solution: The contour curves of the function f are graphs of the equations

   $$x^2 - y^2 = c, \quad \text{for } c \in \mathbb{R}.$$

   These are the lines $y = \mp x$, for $c = 0$, and hyperbolas with asymptotes $y = \mp x$, for $c > 0$. Some of these are sketched in Figure 4. □

4. Give the formula for a linear function $f : \mathbb{R}^2 \to \mathbb{R}$ whose graph contains the points $(1, 4, 7)$, $(4, 7, 0)$ and $(0, 4, 7)$. Sketch the graph of f.

   Solution: The equation of the plane is given, in general, by

   $$z = d + ax + by, \quad \text{for } (x, y) \in \mathbb{R}^2. \tag{1}$$

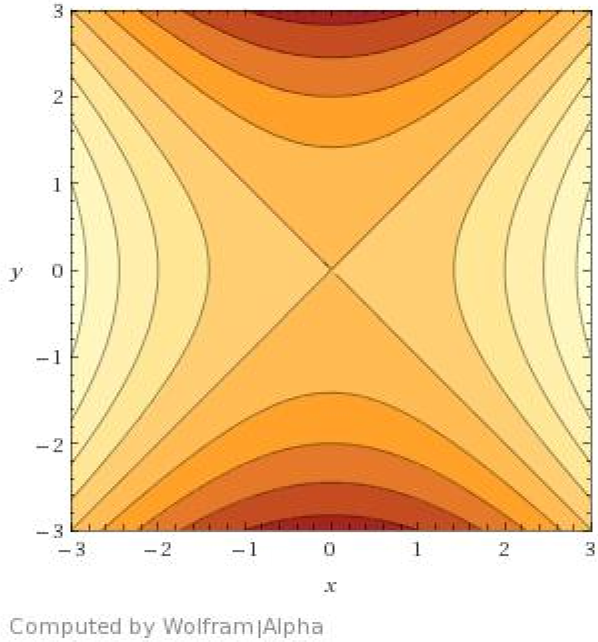

Figure 4: Sketch of Contour Diagram in Problem 3

Since we want the point ((1, 4, 7) to be on the plane determined by (1), we must have that

$$7 = d + a(1) + b(4),$$

or

$$a + 4b + d = 7. \qquad (2)$$

Similarly, for the other two points we get

$$4a + 7b + d = 0, \qquad (3)$$

and

$$4b + d = 7. \qquad (4)$$

It follows from the equations in (2) and (4) that

$$a = 0. \qquad (5)$$

Thus, (3) and (4) become the system

$$\begin{cases} 7b + d = 0 \\ 4b + d = 7 \end{cases}$$

which can be solved to yield

$$b = -\frac{7}{3} \quad \text{and} \quad d = \frac{49}{3}. \tag{6}$$

Putting together the information in (5) and (6), we get from (1) that the equation of the plane through the points $(1, 4, 7)$, $(4, 7, 0)$ and $(0, 4, 7)$ is

$$z = \frac{49}{3} - \frac{7}{3}y.$$

A sketch of the graph of this plane is shown in Figure 5. □

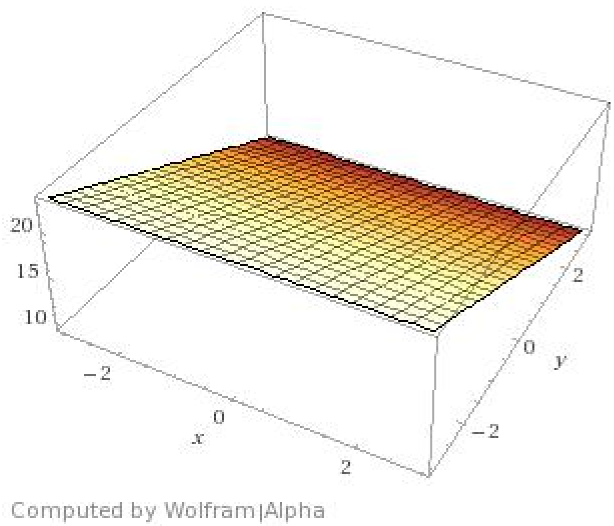

Figure 5: Sketch of Plane in Problem 4

5. Give the equation of plane parallel to the plane $2x + 4y - 3z = 1$ and which goes through the point $(1, 0, -1)$.

   ***Solution**:* The equation of the plane is given, in general, by

   $$z = z_o + a(x - x_o) + b(y - y_o), \quad \text{for } (x, y) \in \mathbb{R}^2. \tag{7}$$

   Since we want the plane to be parallel to

   $$z = -\frac{1}{3} + \frac{2}{3}x + \frac{4}{3}y,$$

it must be the case that

$$a = \frac{2}{3} \quad \text{and} \quad b = \frac{4}{3}. \tag{8}$$

Combining the information in (8) with the requirement that the plane goes through the point $(1, 0, -1)$, we obtain from (7) that the equation of the plane is

$$z = -1 + \frac{2}{3}(x - 1) + \frac{4}{3}y, \quad \text{for } (x, y) \in \mathbb{R}^2,$$

or

$$z = -\frac{5}{3} + \frac{2}{3}x + \frac{4}{3}y, \quad \text{for } (x, y) \in \mathbb{R}^2.$$

□

6. Compute the first partial derivatives of $f \colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = (4x - x^7 - y)^4$ for all $(x, y) \in \mathbb{R}^2$.

   Answer:

$$\frac{\partial f}{\partial x}(x, y) = 4(4 - 7x^6)(4x - x^7 - y)^3, \quad \text{for all } (x, y) \in \mathbb{R}^2,$$

   and

$$\frac{\partial f}{\partial y}(x, y) = -4(4x - x^7 - y)^3, \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

□

7. Give the equation of the tangent plane to the graph of $z = ye^{x/y}$ at the point $(1, 1, e)$.

   Answer: The equation of the tangent plane is

$$z = ex.$$

□

8. Compute the differential of f, where $f(x, y) = \sqrt{x^2 + y^3}$, for all $(x, y) \in \mathbb{R}^2$, at the point $(1, 2)$, and use it to estimate $f(1.04, 1.98)$.

   Solution: The differential of f at the point $(1, 2)$ is

$$df(1, 2) = \frac{\partial f}{\partial x}(1, 2)dx + \frac{\partial f}{\partial y}(1, 2)dy,$$

where
$$\frac{\partial f}{\partial x}(x,y) = \frac{x}{\sqrt{x^2+y^3}} \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = \frac{3}{2}\frac{y^2}{\sqrt{x^2+y^3}},$$
for $(x,y) \neq (0,0)$. Thus,
$$df(1,2) = \frac{1}{3}\ dx + 2\ dy.$$

Next, we estimate
$$f(1.04, 1.98) \quad \approx \quad f(1,2) + df(1,2),$$
where
$$df(1,2) \approx \frac{1}{3}(0.04) + 2(-0.02) \approx -0.03.$$
Thus,
$$f(2.98, 4.01) \approx 3 - 0.03 = 2.97.$$

□

9. Assume that the temperature, $T(x,y)$, at a point (x,y) in the plane is given by
$$T(x,y) = \frac{100}{1+x^2+y^2}, \qquad \text{for all } (x,y) \in \mathbb{R}^2.$$
   (a) Sketch the contour plot for T.
   (b) Locate the hottest point in the plane. What is the temperature at that point?
   (c) Give the direction of greatest increase in temperature at the point $(3,2)$. What is the rate of change of temperature in that direction?
   (d) A bug moves in the plane along a path given by $\overrightarrow{r}(t) = t\widehat{i} + t^2\widehat{j}$ for $t \in \mathbb{R}$. How fast is the temperature changing when $t = 1$.

10. Let $f(x,y) = x^2 + y^2$ for all $(x,y) \in \mathbb{R}^2$. Sketch the flow of the vector field $\nabla F(x,y)$.