Assignment #12

Due on Friday, March 10, 2017

Read Section 5.1 on *Existence and Uniqueness for General Systems* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`.

Read Section 2.6 on *Existence and Uniqueness for Systems* in Blanchard, Devaney and Hall.

Do the following problems

1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$ be differentiable functions whose partial derivatives are continuous for all $(x, y) \in \mathbb{R}^2$, and consider the two–dimensional, autonomous system

$$\begin{cases} \dfrac{dx}{dt} = f(x,y); \\ \\ \dfrac{dy}{dt} = g(x,y). \end{cases} \tag{1}$$

   Let $(x(t), y(t)$, for $t \in \mathbb{R}$, denote a solution of the system in (1). For $\tau \in \mathbb{R}$, define $(u(t), v(t)) = (x(\tau + t), y(\tau + t))$ for all $t \in \mathbb{R}$. Show that (u, v) is also a solution of the the system (1).

2. Let f and g be as in Problem 1 and assume that solutions of the system in (1) exist for all $t \in \mathbb{R}$.

   Apply the local existence and uniqueness theorem to show that distinct trajectories in the phase portrait of the system in (1) do not intersect.

3. Let f and g be as in Problem 1 and assume that solutions of the system in (1) exist for all $t \in \mathbb{R}$. Assume also that $f(0, 0) = 0$ and $g(0, 0) = 0$. Prove that if $(x(t), y(t))$ is a solution of the IVP

$$\begin{cases} \dot{x} = f(x,y); \\ \dot{y} = g(x,y); \\ \\ x(0) = y(0) = 0, \end{cases}$$

   then, $(x(t), y(t)) = (0, 0)$ for all $t \in \mathbb{R}$.

4. Let $g\colon \mathbb{R} \to \mathbb{R}$ denote differentiable function with continuous derivative $g'\colon \mathbb{R} \to \mathbb{R}$, and let $f\colon \mathbb{R} \to \mathbb{R}$ and $b\colon \mathbb{R} \to \mathbb{R}$ denote continuous functions. Apply the local existence and uniqueness theorem for systems to prove that the IVP

$$\begin{cases} \dfrac{d^2x}{dt^2} + b(t)\dfrac{dx}{dt} + g(x) = f(t); \\ \\ x(0) = x_o; \\ x'(0) = y_o, \end{cases}$$

has a unique solutions defined in some interval $(-\delta, \delta)$ around 0, for some $\delta > 0$.

5. Let $g\colon \mathbb{R} \to \mathbb{R}$ denote differentiable function with continuous derivative $g'\colon \mathbb{R} \to \mathbb{R}$. Assume also that $g(0) = 0$.

   Prove that, if $x\colon \Omega \to \mathbb{R}$, for some interval of real numbers, Ω, that contains 0, is a solution of the IVP

$$\begin{cases} \dfrac{d^2x}{dt^2} + g(x) = 0; \\ \\ x(0) = 0; \\ x'(0) = 0, \end{cases}$$

   then $x(t) = 0$ for all $t \in \mathbb{R}$.