

Assignment #16

Due on Friday, April 21, 2017

Read Section 6.3 on *Analysis of a Lotka–Volterra System* in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>

Read Section 5.3 on *Hamiltonian Systems* in Blanchard, Devaney and Hall.

Do the following problems

1. Consider the system

$$\begin{cases} \dot{x} = y; \\ \dot{y} = x^3 - x. \end{cases} \quad (1)$$

- (a) Verify that the function $H: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$H(x, y) = \frac{y^2}{2} + \frac{x^2}{2} - \frac{x^4}{4}, \quad \text{for all } (x, y) \in \mathbb{R}^2, \quad (2)$$

is a conserved quantity of the system in (1).

- (b) Sketch the level sets of the function H given in (2).
(c) Sketch the phase portrait of the system in (1). Determine the nature of the stability of the equilibrium points.

2. For the system

$$\begin{cases} \dot{x} = y; \\ \dot{y} = x - x^2, \end{cases} \quad (3)$$

- (a) find a conserved quantity, H ;
(b) sketch the level sets of the function H found in part (a);
(c) sketch the phase portrait of the system in (3), and determine the nature of the stability of the equilibrium points.

3. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions with antiderivatives F and G , respectively.

Give a conserved quantity for the system

$$\begin{cases} \dot{x} = f(y); \\ \dot{y} = g(x), \end{cases}$$

in terms of the functions F and G .

4. For the Lotka–Volterra system

$$\begin{cases} \dot{x} &= x(1 - y); \\ \dot{y} &= y(x - 1), \end{cases}$$

let $(x(t), y(t))$ be a parametrization of a closed orbit in the first quadrant with period T . Verify that

$$\frac{1}{T} \int_0^T x(t) dt = \frac{1}{T} \int_0^T y(t) dt = 1.$$

Generalize this result for the case of the system

$$\begin{cases} \dot{x} &= \alpha x - \beta xy; \\ \dot{y} &= \delta xy - \gamma y, \end{cases} \quad (4)$$

where α , β , γ and δ are positive parameters.

5. Let α , β , γ and δ be the positive parameters in system (4) in Problem 4, and denote by Q_1^+ the set

$$Q_1^+ = \{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y > 0\}$$

Define $H: Q_1^+ \rightarrow \mathbb{R}$ to be

$$H(x, y) = \delta x - \gamma \ln(x) + \beta y - \alpha \ln(y), \quad \text{for } (x, y) \in Q_1^+. \quad (5)$$

- (a) Verify the function H given (5) is a conserved quantity for the system (4) in Problem 4.
- (b) Show that H in (5) has a unique critical point in Q_1^+ and show that it H attains its minimum in Q_1^+ at that point.
- (c) Sketch the level sets of the function H given in (5).
- (d) Sketch the phase portrait of the system in (4). Determine the nature of the stability of the equilibrium points.