Assignment #7

Due on Friday, February 24, 2017

Read Section 4.1.5 on *Non–Diagonalizable Systems with No Real Eigenvalues* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 3.4 on *Complex Eigenvalues* in Blanchard, Devaney and Hall.

Background and Definitions.
Some Facts from the Arithmetic of Complex Numbers.
The object $z = a + bi$, where a and b are real numbers, denotes a complex number; we write $z \in \mathbb{C}$. The complex number i has the property that $i^2 = -1$.

- **Real Part of a Complex Number**. The real part of $z = a + bi$, denoted by $\mathrm{Re}(z)$, is defined by
$$\mathrm{Re}(z) = a.$$

- **Imaginary Part of a Complex Number**. The imaginary part of $z = a + bi$, denoted by $\mathrm{Im}(z)$, is defined by
$$\mathrm{Im}(z) = b.$$

- **Conjugate of a Complex Number**. The conjugate of $z = a + bi$, denoted by $\overline{z}$, is defined by
$$\overline{z} = a - bi.$$

  Important Observations:

  $z \in \mathbb{C}$ is a real number if and only if $\overline{z} = z$.
  $\mathrm{Re}(z) = \dfrac{z + \overline{z}}{2}$ and $\mathrm{Im}(z) = \dfrac{z - \overline{z}}{2i}$.
  For any complex numbers z and w, $\overline{zw} = \overline{z}\ \overline{w}$.

- **Modulus of a Complex Number**. The modulus of $z = a + bi$, denoted by $|z|$, is defined by
$$|z| = \sqrt{a^2 + b^2}.$$

  Note that $z\overline{z} = |z|^2$.

- **Argument of a Complex Number**. The argument of $z = a + bi$, denoted by $\arg(z)$, is defined by
$$\arg(z) = \arctan\left(\frac{b}{a}\right).$$

Do the following problems

1. Let A denote a 2×2 matrix with real entries. Assume that $\lambda \in \mathbb{C}$ is a complex eigenvalue of A with (complex) eigenvector w; that is, $w = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, where $z_1, z_2 \in \mathbb{C}$. Show that $\overline{\lambda}$ is also an eigenvalue of A with corresponding eigenvector $\overline{w} = \begin{pmatrix} \overline{z_1} \\ \overline{z_2} \end{pmatrix}$.

2. Let A denote a 2×2 matrix with real entries. Assume that $\lambda = \alpha + i\beta$ is a complex eigenvalue of A, where $\beta \neq 0$. Show that the characteristic polynomial of A is given by $p_A(\lambda) = \lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2$.

3. Let A denote a 2×2 matrix with real entries with eigenvalues $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, with $\beta \neq 0$. Let $w_1 \in \mathbb{C}^2$ be an eigenvector corresponding to λ_1.

   (a) Show that $w_2 = \overline{w_1}$ is an eigenvector corresponding to λ_2.

   (b) Define vectors $v_1, v_2 \in \mathbb{R}^2$ by
   $$v_1 = \text{Im}(w_1) = \frac{1}{2i}(w_1 - w_2) \quad \text{and} \quad v_2 = \text{Re}(w_1) = \frac{1}{2}(w_1 + w_2).$$
   Show that the set $\{v_1, v_2\}$ is linearly independent.

   (c) Verify that
   $$\begin{array}{rcl} Av_1 & = & \alpha v_1 + \beta v_2 \\ Av_2 & = & -\beta v_1 + \alpha v_2 \end{array} \tag{1}$$

4. Let A, $\lambda_1 = \alpha + i\beta$, v_1 and v_2 be as in Problem 3.

   (a) Use the result in (1) to deduce that the matrix representation of the linear transformation induced by A relative to the basis $\mathcal{B} = \{v_1, v_2\}$ is
   $$[A]_\mathcal{B}^\mathcal{B} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

   (b) Give an invertible matrix Q such that $Q^{-1}AQ = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$.

5. Construct solutions of the system
   $$\begin{cases} \dot{x} & = & ay; \\ \dot{y} & = & -bx, \end{cases}$$
   where a and b are positive constants, and sketch the phase portrait.