Review Problems for Exam 2

1. Compute the fundamental matrix for the system

$$\begin{cases} \dot{x} = -3x - y; \\ \dot{y} = \ \ 4x - 3y. \end{cases}$$

Give the general solution of the system and determine the nature of the stability of the equilibrium point $(0, 0)$.

2. Compute the general solution of the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

and describe the nature of the stability of its equilibrium point. Sketch the phase portrait.

3. Give the general solution of the system

$$\begin{cases} \dot{x} = 2x + y + 1; \\ \dot{y} = x - 2y - 1. \end{cases}$$

Determine the nature of the stability of the equilibrium point of the system. Sketch the phase portrait.

4. Let A denote the 2×2 matrix, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b and c are real numbers, and consider the linear system of differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}. \tag{1}$$

Let $E_A(t)$, for $t \in \mathbb{R}$, denote the fundamental matrix of the system in (1).

(a) Put $W(t) = \det(E_A(t))$, for all $t \in \mathbb{R}$. Verify that W solves the differential equation

$$\frac{dW}{dt} = (\lambda_1 + \lambda_2)W, \qquad \text{for all } t \in \mathbb{R}, \tag{2}$$

where λ_1 and λ_2 are the eigenvalues of A.

(b) Solve the differential equation in (2) to deduce that $W(t) = e^{(\lambda_1+\lambda_2)t}$, for all $t \in \mathbb{R}$. Deduce that the columns of $E_A(t)$ are linearly independent solutions of the system in (1).

5. Find two distinct solutions of the initial value problem

$$\begin{cases} \dot{x} = 6tx^{2/3}; \\ x(0) = 0. \end{cases}$$

Why doesn't this violate the uniqueness assertion of the local existence and uniqueness theorem?

6. Consider the initial value problem $\begin{cases} \dfrac{dy}{dt} = y^2 - y; \\ \\ y(0) = 2 \end{cases}$

Give the maximal interval of existence for the solution. Does the solution exist for all t? If not, explain what prevents the solution from being extended further.

7. The motion of an object of mass m, attached to a spring of stiffness constant k, and moving along a horizontal flat surface is modeled by the second–order, linear differential equation

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0, \tag{3}$$

where $x = x(t)$ denotes the position of the object along its direction of motion, and γ is the coefficient of friction between the object and the surface.

(a) Express the equation in (3) as a system of first order linear differential equations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4}$$

(b) For the matrix A in (4), let $\omega^2 = \dfrac{k}{m}$ and $b = \dfrac{\gamma}{2m}$.
Give the characteristic polynomial of the matrix A, and determine when the A has (i) two real and distinct eigenvalues; (ii) only one real eigenvalue; (iii) complex eigenvalues with nonzero imaginary part.

(c) Describe the behavior of solutions of (3) in case (iii) of part (b)

8. Let Ω denote an open interval of real numbers, and $f\colon \Omega \to \mathbb{R}$ denote a continuous function. Let $x_p\colon \Omega \to \mathbb{R}$ denote a particular solution of the nonhomogeneous, second–order equation

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t), \qquad \text{for } t \in \Omega, \tag{5}$$

where b and c are real constants.

(a) Let $x\colon \Omega \to \mathbb{R}$ denote any solution of (5) and put

$$u(t) = x(t) - x_p(t), \qquad \text{for } t \in \Omega.$$

Verify that u solves the homogeneous, second–order equation

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0, \qquad \text{for } t \in \Omega. \tag{6}$$

(b) Let $x_1\colon \Omega \to \mathbb{R}$ and $x_2\colon \Omega \to \mathbb{R}$ denote linearly independent solutions of the homogenous equation (6). Prove that any solution of the nonhomogeneous equation in (5) must be of the form

$$x(t) = c_1x_1(t) + c_2x_2(t) + x_p(t), \qquad \text{for all } t \in \Omega,$$

where c_1 and c_2 are constants.