Solutions to Review Problems for Exam 2

1. Compute the fundamental matrix for the system

$$\begin{cases} \dot{x} = -3x - y; \\ \dot{y} = \ \ 4x - 3y \end{cases} \tag{1}$$

Give the general solution of the system and determine the nature of the stability of the equilibrium point $(0, 0)$.

Solution: Write the system in vector form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix},$$

where A is the matrix

$$A = \begin{pmatrix} -3 & -1 \\ 4 & -3 \end{pmatrix}. \tag{2}$$

The characteristic polynomial of A in (2) is

$$p_A(\lambda) = \lambda^2 + 6\lambda + 13,$$

which can be written as

$$p_A(\lambda) = (\lambda^2 + 6\lambda + 9) + 4,$$

or

$$p_A(\lambda) = (\lambda + 3)^2 + 4. \tag{3}$$

It follows from (3) that the eigenvalues of A in (2) are

$$\lambda_1 = -3 + 2i \quad \text{and} \quad \lambda_2 = -3 - 2i \tag{4}$$

We look for an invertible matrix Q such that

$$Q^{-1}AQ = J,$$

where

$$J = \begin{pmatrix} -3 & -2 \\ 2 & -3 \end{pmatrix} \tag{5}$$

In order to do this, we first find an eigenvector $w_1 \in \mathbb{C}^2$ corresponding to λ_1 in (4). We get

$$w_1 = \begin{pmatrix} i \\ 2 \end{pmatrix}.$$

Then, set

$$v_1 = \mathrm{Im}(w_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2 = \mathrm{Re}(w_1) = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

and

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}; \tag{6}$$

so that

$$Q^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}. \tag{7}$$

The fundamental matrix, E_J associated with J in (5) is

$$E_J(t) = e^{-3t} \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}. \tag{8}$$

Using (5), (6) and (8), we can compute the fundamental matrix corresponding to A by using

$$E_A(t) = QE_J(t)Q^{-1}, \quad \text{for all } t \in \mathbb{R}.$$

We get

$$E_A(t) = e^{-3t} \begin{pmatrix} \cos 2t & -\frac{1}{2}\sin 2t \\ 2\sin 2t & \cos 2t \end{pmatrix}, \quad \text{for all } t \in \mathbb{R},$$

is the fundamental matrix for the system in (1).

The general solution of the system in (1) is then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = E_A(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants, or

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-3t}\cos 2t - \dfrac{c_2}{2} e^{-3t}\sin 2t \\ 2c_1 e^{-3t}\sin 2t + c_2 e^{-3t}\cos 2t \end{pmatrix}, \quad \text{for all } t \in \mathbb{R},$$

where c_1 and c_2 are arbitrary constants.

Since the eigenvalues of A in (4) are complex with negative real part, $(0,0)$ is a spiral sink. □

2. Compute the general solution of the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{9}$$

and describe the nature of the stability of its equilibrium point. Sketch the phase portrait.

Solution: We first compute the fundamental matrix for the system in (9).

Set
$$A = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}. \tag{10}$$

The characteristic polynomial of A in (10) is
$$p_A(\lambda) = \lambda^2 + 6\lambda + 9,$$
which we can write as
$$p_A(\lambda) = (\lambda + 3)^2.$$

Thus,
$$\lambda = -3 \tag{11}$$
is the only eigenvalue of the matrix A in (10).

Next, we find an eigenvector corresponding to $\lambda = -3$, by solving the homogeneous system
$$(A - \lambda I)\mathrm{v} = \mathbf{0}, \tag{12}$$
with $\lambda = -3$. We get the vector
$$\mathrm{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{13}$$

There is no basis for $\mathbb{R}^2$ made up of eigenvectors of A; therefore, A is not diagonalizable. We therefore need to find a solution, v_2, of the nonhomogeneous system
$$(A - \lambda I)\mathrm{v} = \mathrm{v}_1, \tag{14}$$
with $\lambda = -3$. A solution of (14) is
$$\mathrm{v}_2 = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}. \tag{15}$$

Set $Q = \begin{bmatrix} \mathrm{v}_1 & \mathrm{v}_2 \end{bmatrix}$, where v_1 and v_2 are given in (13) and (15), respectively; so that,
$$Q = \begin{pmatrix} 1 & 1/4 \\ 1 & 0 \end{pmatrix}. \tag{16}$$

Next, set
$$J = Q^{-1}AQ, \tag{17}$$

where

$$Q^{-1} = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix}. \tag{18}$$

It follows from (10), (17), (16) and (18) that

$$J = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix}. \tag{19}$$

The fundamental matrix, $E_J(t)$, corresponding to the matrix J in (19) is given by

$$E_J(t) = \begin{pmatrix} e^{-3t} & te^{-3t} \\ 0 & e^{-3t} \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}. \tag{20}$$

The fundamental matrix corresponding to the matrix A in (10) is then given by

$$E_A(t) = QE_J(t)Q^{-1}, \quad \text{for all } t \in \mathbb{R},$$

where Q, $E_J(t)$ and Q^{-1} are given in (16), (20) and (18), respectively. We obtain

$$E_A(t) = \begin{pmatrix} e^{-3t} + 4te^{-3t} & -4te^{-3t} \\ 4te^{-3t} & e^{-3t} - 4te^{-3t} \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}. \tag{21}$$

The general solution of the system in (9) is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = E_A(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad \text{for all } t \in \mathbb{R},$$

for constants c_1 and c_2, and where $E_A(t)$ is given in (21).

Since the only eigenvalue of A in (11) is negative, it follows that $(0,0)$ is asymptotically stable.

A sketch of the phase portrait of the system in (9) is shown in Figure 1. The sketch also shows the nullclines of the system. □

3. Give the general solution of the system

$$\begin{cases} \dot{x} = 2x + y + 1; \\ \dot{y} = x - 2y - 1. \end{cases} \tag{22}$$

Determine the nature of the stability of the equilibrium point of the system. Sketch the phase portrait.

***Solution**:* Write the system in matrix form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}, \tag{23}$$

where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \tag{24}$$

and

$$\begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \text{for all } t \in \mathbb{R}. \tag{25}$$

Next, compute the fundamental matrix $E_A(t)$ associated with A. To do this, we first compute the characteristic polynomial of A,

$$p_A(\lambda) = \lambda^2 - 5,$$

from which we get that

$$\lambda_1 = -\sqrt{5} \quad \text{and} \quad \lambda_2 = \sqrt{5},$$

are the eigenvalues of A given in (24). Corresponding eigenvectors are

$$\mathrm{v}_1 = \begin{pmatrix} 2 - \sqrt{5} \\ 1 \end{pmatrix} \quad \text{and} \quad \mathrm{v}_2 = \begin{pmatrix} 2 + \sqrt{5} \\ 1 \end{pmatrix}. \tag{26}$$

Thus, setting $Q = [\mathrm{v}_1 \quad \mathrm{v}_2]$, where v_1 and v_2 are as given in (26); so that,

$$Q = \begin{pmatrix} 2 - \sqrt{5} & 2 + \sqrt{5} \\ 1 & 1 \end{pmatrix}, \tag{27}$$

we obtain that

$$Q^{-1} = \begin{pmatrix} -\dfrac{1}{2\sqrt{5}} & \dfrac{1}{2} + \dfrac{1}{\sqrt{5}} \\ \\ \dfrac{1}{2\sqrt{5}} & \dfrac{1}{2} - \dfrac{1}{\sqrt{5}} \end{pmatrix}, \tag{28}$$

and

$$Q^{-1}AQ = J = \begin{pmatrix} -\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix}. \tag{29}$$

Thus, A is diagonalizable.

The fundamental matrix associated with J given in (29) is

$$E_J(t) = \begin{pmatrix} e^{-\sqrt{5}t} & 0 \\ 0 & e^{\sqrt{5}t} \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}.$$

We can use this matrix to obtain the fundamental matrix associated with A,

$$E_A(t) = QE_J(t)Q^{-1},$$

where Q and Q^{-1} are given in (27) and (28), respectively, to obtain

$$E_A(t) = \begin{pmatrix} \left(\frac{1}{2} - \frac{1}{\sqrt{5}}\right) e^{-\sqrt{5}t} + \left(\frac{1}{2} + \frac{1}{\sqrt{5}}\right) e^{\sqrt{5}t} & -\frac{1}{2\sqrt{5}}e^{-\sqrt{5}t} + \frac{1}{2\sqrt{5}}e^{\sqrt{5}t} \\ -\frac{1}{2\sqrt{5}}e^{-\sqrt{5}t} + \frac{1}{2\sqrt{5}}e^{\sqrt{5}t} & \left(\frac{1}{2} + \frac{1}{\sqrt{5}}\right) e^{-\sqrt{5}t} + \left(\frac{1}{2} - \frac{1}{\sqrt{5}}\right) e^{\sqrt{5}t} \end{pmatrix}, \tag{30}$$

for $t \in \mathbb{R}$.

The general solution of the system in (23) is then given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = E_A(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + E_A(t) \int_0^t E_A(-\tau) \begin{pmatrix} b_1(\tau) \\ b_2(\tau) \end{pmatrix} \, d\tau, \quad \text{for } t \in \mathbb{R}, \tag{31}$$

where c_1 and c_2 are arbitrary constants, and the vector–values function $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is given in (25).

We first determine the nullclines:

$$\begin{array}{ll} \dot{x} = 0\text{–nullcline:} & 2x + y = -1 \\ \dot{y} = 0\text{–nullcline:} & x - 2y = 1 \end{array}$$

These lines are sketched in Figure 2. The nullclines intersect at the equilibrium point

$$(\overline{x}, \overline{y}) = \left(-\frac{1}{5}, -\frac{3}{5}\right). \tag{32}$$

Since the eigenvalues of A are

$$\lambda = \pm\sqrt{5},$$

$(\overline{x}, \overline{y})$ is a saddle point for the system in (22). A sketch of the phase portrait of the system in (22) is shown in Figure 2. □

4. Let A denote the 2×2 matrix, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b and c are real numbers, and consider the linear system of differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}. \tag{33}$$

Let $E_{_A}(t)$, for $t \in \mathbb{R}$, denote the fundamental matrix of the system in (33).

(a) Put $W(t) = \det(E_{_A}(t))$, for all $t \in \mathbb{R}$. Verify that W solves the differential equation

$$\frac{dW}{dt} = (\lambda_1 + \lambda_2)W, \quad \text{for all } t \in \mathbb{R}, \tag{34}$$

where λ_1 and λ_2 are the eigenvalues of A.

***Solution**:* Write

$$E_{_A}(t) = \begin{pmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{pmatrix}, \quad \text{for } t \in \mathbb{R}, \tag{35}$$

where the vector–valued functions

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

are solutions of the system in (33),

$$\begin{pmatrix} x_1(0) \\ y_1(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_2(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{36}$$

We then have that

$$\begin{cases} x_1'(t) = ax_1(t) + by_1(t); \\ y_1'(t) = cx_1(t) + dy_1(t), \end{cases} \quad \text{for } t \in \mathbb{R}, \tag{37}$$

and

$$\begin{cases} x_2'(t) = ax_2(t) + by_2(t); \\ y_2'(t) = cx_2(t) + dy_2(t), \end{cases} \quad \text{for } t \in \mathbb{R}. \tag{38}$$

Now, since $W(t) = \det(E_{_A}(t))$, for all $t \in \mathbb{R}$, it follows from (35) that

$$W(t) = x_1(t)y_2(t) - x_2(t)y_1(t), \quad \text{for all } t \in \mathbb{R}. \tag{39}$$

Differentiating on both sides of (39), and using the product rule, we obtain

$$W'(t) = x_1'(t)y_2(t) + x_1(t)y_2'(t) - x_2'(t)y_1(t) - x_2(t)y_1'(t), \quad \text{for } t \in \mathbb{R};$$

so that, using (37) and (38),

$$\frac{dW}{dt} = (ax_1 + by_1)y_2 + x_1(cx_2 + dy_2) - (ax_2 + by_2)y_1 - x_2(cx_1 + dy_1);$$

so that,

$$\frac{dW}{dt} = ax_1y_2 + by_1y_2 + cx_1x_2 + dx_1y_2 - ax_2y_1 - by_2y_1 - cx_1x_2 - dx_2y_1,$$

or

$$\begin{aligned}\frac{dW}{dt} &= ax_1y_2 + dx_1y_2 - ax_2y_1 - dx_2y_1 \\ &= a(x_1y_2 - x_2y_1) + d(x_1y_2 - x_2y_1);\end{aligned}$$

hence, using (39),

$$\frac{dW}{dt} = (a+d)W,$$

or

$$\frac{dW}{dt} = \text{trace}(A)W. \tag{40}$$

Thus, since $\text{trace}(A) = \lambda_1 + \lambda_2$, (34) follows from (40). □

(b) Solve the differential equation in (34) to deduce that $W(t) = e^{(\lambda_1+\lambda_2)t}$, for all $t \in \mathbb{R}$. Deduce that the columns of $E_A(t)$ are linearly independent solutions of the system in (33).

***Solution**:* It follows from (35), the definition of W and (36) that

$$W(0) = \det(I) = 1.$$

Thus, according to the result of part (a), W is the solution of the IVP

$$\begin{cases} \dfrac{dW}{dt} = (\lambda_1 + \lambda_2)W; \\ W(0) = 1. \end{cases} \tag{41}$$

The IVP (41) has a unique solution given by $W(t) = e^{(\lambda_1+\lambda_2)t}$, for all $t \in \mathbb{R}$. It then follows that $W(t) \neq 0$ for all $t \in \mathbb{R}$. Thus, the columns of $E_A(t)$ are linearly independent. □

5. Find two distinct solutions of the initial value problem

$$\begin{cases} \dot{x} = 6tx^{2/3}; \\ x(0) = 0. \end{cases} \tag{42}$$

Why doesn't this violate the uniqueness assertion of the local existence and uniqueness theorem?

Solution: Use separation of variables to show that the function

$$x_1(t) = t^6, \quad \text{for all } t \in \mathbb{R},$$

solves the initial value problem (IVP) in (42).

Verify that the function

$$x_2(t) = 0, \quad \text{for all } t \in \mathbb{R},$$

also solves the IVP in (42).

Thus, the IVP in (42) has at least two distinct solutions.

Observe that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,t) = 6tx^{2/3}, \quad \text{for } (x,t) \in \mathbb{R}^2,$$

does not have a continuous partial derivative with respect to x at $(0,0)$. Indeed, for $t \neq 0$ and $x \neq 0$,

$$\frac{\partial f}{\partial x} = \frac{4t}{x^{1/3}}$$

does not have a limit as (x,t) approaches $(0,0)$. Hence, the local existence and uniqueness theorem discussed in class does not apply to the IVP (42). □

6. Consider the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = y^2 - y; \\ \\ y(0) = 2. \end{cases} \tag{43}$$

Give the maximal interval of existence for the solution. Does the solution exist for all t? If not, explain what prevents the solution from being extended further.

Solution: Use separation of variables and partial fractions to derive the solution

$$y(t) = \frac{2}{2 - e^t}, \quad \text{for } t < \ln(2). \tag{44}$$

Note that the denominator of the expression in (44) is 0 when $t = \ln(2)$. At that time the solution of the IVP in (43) given in (44) ceases to exist. Hence, the maximal interval of existence for the solution of the IVP in (43) is $(-\infty, \ln(2))$. □

7. The motion of an object of mass m, attached to a spring of stiffness constant k, and moving along a horizontal flat surface is modeled by the second–order, linear differential equation

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0, \tag{45}$$

where $x = x(t)$ denotes the position of the object along its direction of motion, and γ is the coefficient of friction between the object and the surface.

(a) Express the equation in (45) as a system of first order linear differential equations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}. \tag{46}$$

***Solution**:* The matrix A in (46) is given by

$$A = \begin{pmatrix} 0 & 1 \\ -k/m & -\gamma/m \end{pmatrix}. \tag{47}$$

□

(b) For the matrix A in (46), let $\omega^2 = \dfrac{k}{m}$ and $b = \dfrac{\gamma}{2m}$.
Give the characteristic polynomial of the matrix A, and determine when the A has (i) two real and distinct eigenvalues; (ii) only one real eigenvalue; (iii) complex eigenvalues with nonzero imaginary part.
***Solution**:* The matrix A in (47) can now be written as

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2b \end{pmatrix}. \tag{48}$$

The characteristic polynomial of the matrix A in (48) is then

$$p_{_A}(\lambda) = \lambda^2 + 2b\lambda + \omega^2.$$

Thus, the eigenvalues of the matrix A in (48) are given by

$$\lambda = -b \pm \sqrt{b^2 - \omega^2}.$$

Thus, A has

(i) two real and distinct eigenvalues, if $b > \omega$;
(ii) only one real eigenvalue, if $b = \omega$;

(iii) complex eigenvalues with nonzero imaginary part, if $b < \omega$.

□

(c) Describe the behavior of solutions of (45) in case (iii) of part (b).

Solution: If $b < \omega$, the eigenvalues of A are complex with negative real part. Hence, the solutions of the equation (45) will oscillate with decreasing amplitude. □

8. Let Ω denote an open interval of real numbers, and $f\colon \Omega \to \mathbb{R}$ denote a continuous function. Let $x_p\colon \Omega \to \mathbb{R}$ denote a particular solution of the nonhomogeneous, second–order equation

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t), \quad \text{for } t \in \Omega, \tag{49}$$

where b and c are real constants.

(a) Let $x\colon \Omega \to \mathbb{R}$ denote any solution of (49) and put

$$u(t) = x(t) - x_p(t), \quad \text{for } t \in \Omega.$$

Verify that u solves the homogeneous, second–order equation

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0, \quad \text{for } t \in \Omega. \tag{50}$$

Solution: Let $x\colon \Omega \to \mathbb{R}$ be a solution of (49). Then,

$$x''(t) + bx'(t) + cx(t) = f(t), \quad \text{for } t \in \Omega. \tag{51}$$

Since we are assuming the $x_p\colon \Omega\mathbb{R}$ also solves (49), we also have that

$$x_p''(t) + bx_p'(t) + cx_p(t) = f(t), \quad \text{for } t \in \Omega. \tag{52}$$

Put

$$u(t) = x(t) - x_p(t), \quad \text{for } t \in \Omega. \tag{53}$$

Then, by properties of differentiation,

$$u'(t) = x'(t) - x_p'(t), \quad \text{for } t \in \Omega,$$

and

$$u''(t) = x''(t) - x_p''(t), \quad \text{for } t \in \Omega.$$

Thus, using (51), (52) and (53), we have that

$$\begin{aligned} u''(t) + bu'(t) + cu(t) &= x''(t) - x_p''(t) + b(x'(t) - x_p'(t)) + c(x(t) - x_p(t)) \\ &= x''(t) - x_p''(t) + bx'(t) - bx_p'(t) + cx(t) - cx_p(t) \\ &= x''(t) + bx'(t) + cx(t) - (x_p''(t) + bx_p'(t) + cx_p(t)) \\ &= f(t) - f(t), \end{aligned}$$

for all $t \in \Omega$; so that

$$u''(t) + bu'(t) + cu(t) = 0, \quad \text{for all } t \in \Omega,$$

which shows that u solves the equation in (50). □

(b) Let $x_1 \colon \Omega \to \mathbb{R}$ and $x_2 \colon \Omega \to \mathbb{R}$ denote linearly independent solutions of the homogenous equation (50). Prove that any solution of the nonhomogeneous equation in (49) must be of the form

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + x_p(t), \quad \text{for all } t \in \Omega,$$

where c_1 and c_2 are constants.

***Solution**:* Since x_1 and x_2 are linearly independent solutions of (50), any solution of (50) is a linear combination of x_1 and x_2 by the results of problem 4 in Assignment 11. We then have that

$$u(t) = c_1 x_1(t) + c_2 x_2(t), \quad \text{for all } t \in \Omega,$$

for u given in (53) in part (a) of this problem. Consequently,

$$x(t) - x_p(t) = c_1 x_1(t) + c_2 x_2(t), \quad \text{for all } t \in \Omega,$$

fromw which we get that

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + x_p(t), \quad \text{for all } t \in \Omega,$$

which was to be shown. □

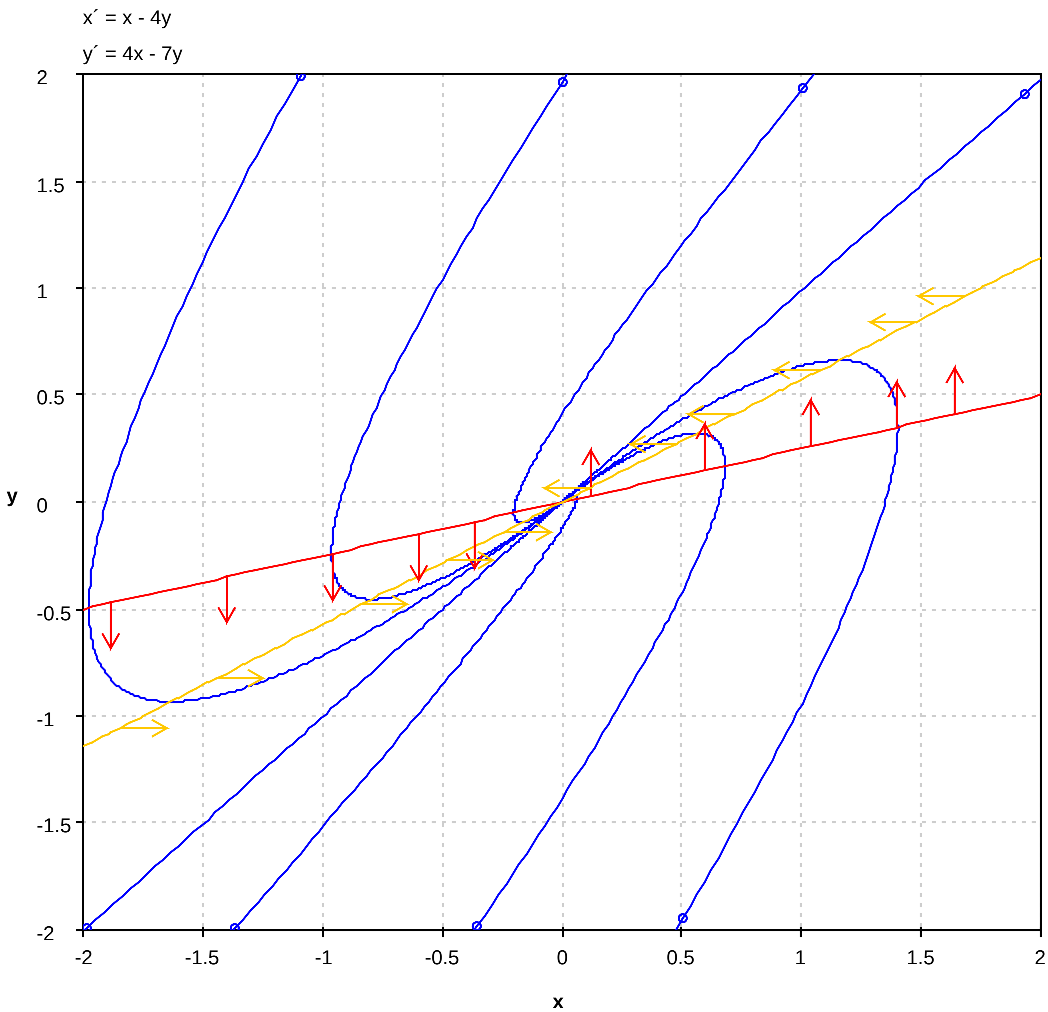

Figure 1: Sketch of Phase Portrait for System (9)

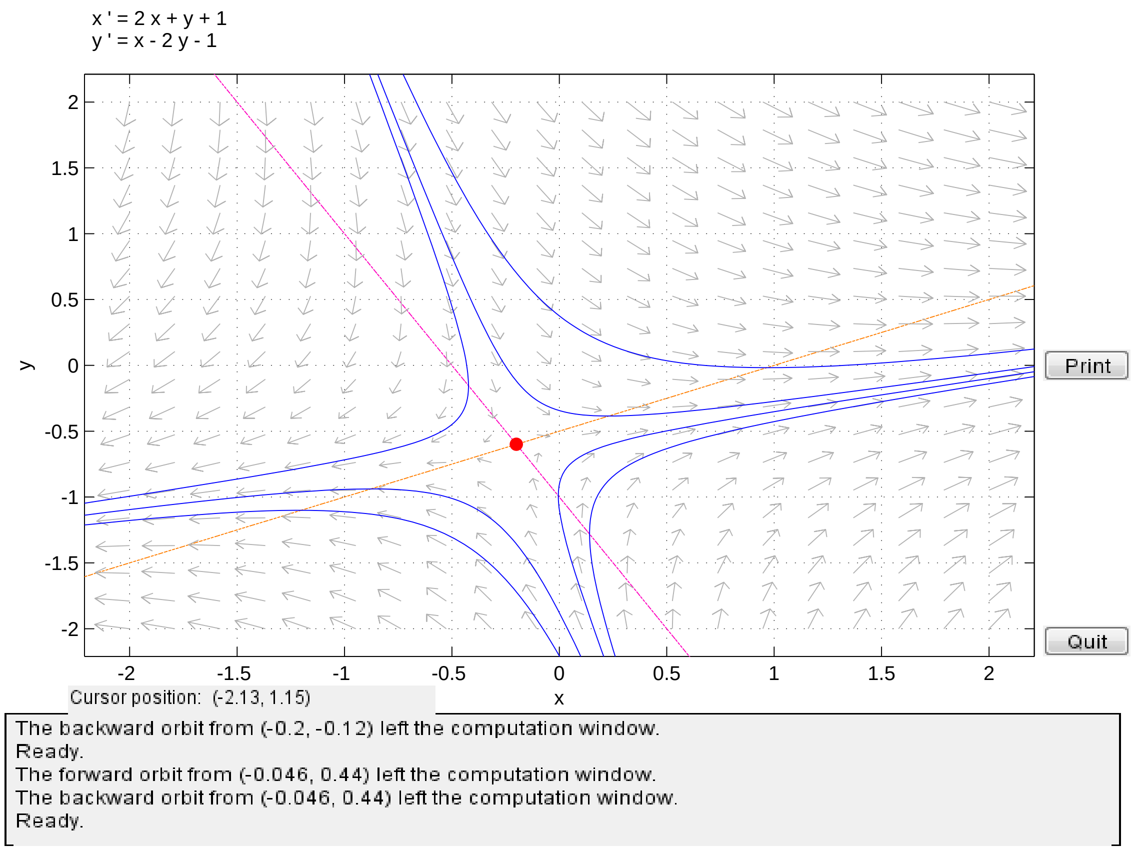

Figure 2: Sketch of Phase Portrait for System (22)