Assignment #1

Due on Wednesday, January 24, 2018

Read Chapter 1 in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Chapter 2 on *Introduction to Modeling* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 1.1 on *Modeling via Differential Equations* in Blanchard, Devaney and Hall.

Do the following problems

1. The diagram in Figure 1 shows a simplification of the chemostat model discussed in Section 2.2 in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`.

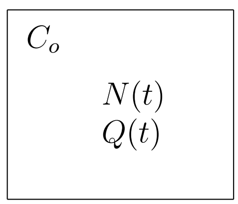

Figure 1: One–Compartment Model

The compartment in the diagram in Figure 1 represents a culture chamber containing $N(t)$ bacteria and a quantity $Q(t)$ of nutrient at time t. The quantities N and Q are assumed to be differentiable functions of t. Assume also that there is no flow of culture in or out of the chamber and that the culture in the chamber is kept well–stirred. In addition, assume that there is an initial amount of nutrient, Q_o, at an initial concentration of C_o, and that there are N_o bacteria at time $t = 0$. Postulate that the *per–capita* growth, $K(c)$, is a function of the nutrient concentration,

$$c(t) = \frac{Q(t)}{V},$$

where V is the volume of the culture, which is assumed to be constant. Assuming that $Y = 1/\alpha$ new cells are produced as s result of consumption of one unit of nutrient, apply conservations principles to obtain a model for the evolution of N and Q in the chamber.

2. Combine the differential equations derived in Problem 1 to show that

$$\frac{d}{dt}[\alpha N + Q] = 0.$$

Deduce therefore that

$$\alpha N(t) + Q(t) = \alpha N_o + Q_o, \quad \text{for all } t.$$

3. Denote $\alpha N_o + Q_o$ by A_o and use the result in Problems 2 to obtain the formula

$$c = \frac{A_o}{V} - \frac{\alpha}{V}N \tag{1}$$

for the concentration of nutrient.

(a) Give an interpretation for the expression in (1).

(b) Denote A_o/V by c_o. Explain why c_o is the nutrient concentration in the absence of bacteria.

4. Assume the constitutive equation $K(c) = mc$, where m is a positive constant of proportionality. Combine the results in Problems 1 and 3 to derive the differential equation

$$\frac{dN}{dt} = mN\left(c_o - \frac{\alpha}{V}N\right). \tag{2}$$

5. Set $r = mc_o$ and $L = \dfrac{c_oV}{\alpha}$, and use (2) to derive a single equation describing the growth of bacteria in the chamber. This equation is known as the Logistic Equation.