

Assignment #13

Due on Monday, April 9, 2018

Read Section 6.1 on *Nondimensionalization* in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>

Read Section 5.1 on *Equilibrium Point Analysis* in Blanchard, Devaney and Hall.

Read Section 5.2 on *Qualitative Analysis* in Blanchard, Devaney and Hall.

Background and Definitions.

In Section 6.1 in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>, we derived the following system of ordinary differential equations for the chemostat system,

$$\begin{cases} \frac{dn}{dt} = \frac{bnc}{a+c} - \frac{F}{V}n; \\ \frac{dc}{dt} = \frac{F}{V}c_o - \frac{F}{V}c - \frac{\alpha bnc}{a+c}. \end{cases} \quad (1)$$

The variables n and c are the bacterial population density and nutrient concentration, respectively, in the chemostat; these are assumed to be differentiable functions of time, t . The parameter c_o , F , V , α , a and b have the following interpretations:

- c_o is the nutrient concentration in a reservoir that feeds the chemostat chamber at a constant rate F ;
- F is also the rate at which culture is drawn from the chemostat chamber;
- V is the fixed volume of the culture;
- α is related to the yield, $Y = 1/\alpha$, which is the number of new cells produced in the chemostat due to consumption of one unit of nutrient;
- b is the maximum *per-capita* growth rate allowed by the medium, and a is the nutrient concentration at which the *per-capita* growth rate is $b/2$.

Do the following problems

1. Introduce new dimensionless variables

$$\hat{n} = \frac{n}{\mu}, \quad \hat{c} = \frac{c}{a}, \quad \text{and} \quad \tau = \frac{t}{\lambda}, \quad (2)$$

where μ and λ are scaling parameters having units of cells/volume and time, respectively.

Verify that the second equation in the system in (1) can be written in the form

$$\frac{d\hat{c}}{d\tau} = \alpha_2 - \frac{\hat{n}\hat{c}}{1 + \hat{c}} - \hat{c},$$

where

$$\alpha_2 = \frac{c_o}{a} \tag{3}$$

and

$$\mu = \frac{a}{\alpha b \lambda}. \tag{4}$$

2. Verify that the parameter α_2 in (3) is dimensionless and that the units of μ defined in (4) are indeed cells/volume. Justify your answers.
3. In Section 6.1 in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>, the system in (1) was nondimensionalized to yield the system

$$\begin{cases} \frac{d\hat{n}}{d\tau} = \alpha_1 \frac{\hat{n}\hat{c}}{1 + \hat{c}} - \hat{n}; \\ \frac{d\hat{c}}{d\tau} = \alpha_2 - \frac{\hat{n}\hat{c}}{1 + \hat{c}} - \hat{c}. \end{cases} \tag{5}$$

- (a) Compute the equilibrium solutions of the system in (5) in the $\hat{n}\hat{c}$ -phase plane.
 - (b) Give interpretations for each of the equilibrium points obtained in part (a). Give conditions under which the system in (5) yields biologically feasible equilibrium solutions.
4. Put

$$F(\hat{n}, \hat{c}) = \begin{pmatrix} \alpha_1 \frac{\hat{n}\hat{c}}{1 + \hat{c}} - \hat{n} \\ \alpha_2 - \frac{\hat{n}\hat{c}}{1 + \hat{c}} - \hat{c} \end{pmatrix}$$

Compute the derivative matrix, $DF(\hat{n}, \hat{c})$, of F .

5. Compute the eigenvalues of $DF(\hat{n}, \hat{c})$ at the equilibrium points found in Problem 3 and use this information to determine their stability properties. What do you conclude about the chemostat system?