Solutions to Review Problems for Exam 3

1. Find the equilibrium solutions of following autonomous differential equations and determine the nature of the satiability of the equilibrium solutions. Sketch some possible solution curves. If possible, describe the long–term behavior of the solutions.

   (a) $\dfrac{dx}{dt} = (x-3)(x-5)$

   ***Solution**:* Set $f(x) = (x-3)(x-5)$, for all $x \in \mathbb{R}$, or $f(x) = x^2 - 8x + 15$, for all $x \in \mathbb{R}$. We would like to analyze the ODE

$$\frac{dx}{dt} = f(x), \tag{1}$$

   where the graph of f is sketched in Figure 1. We see from the sketch that

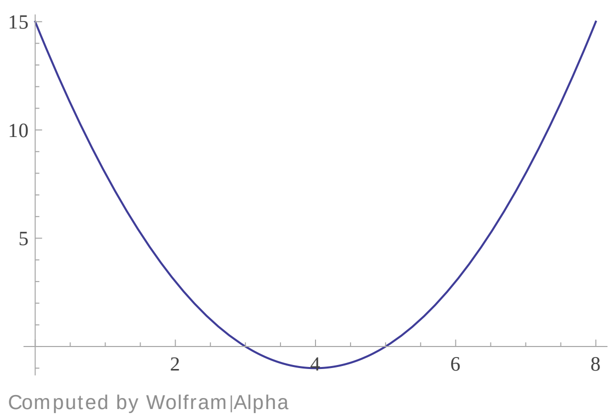

Figure 1: Sketch of $f(x)$ versus x

   the ODE in (1) has equilibrium points at

$$\overline{x}_1 = 3 \quad \text{and} \quad \overline{x}_2 = 5.$$

   We can also see from the sketch of $f(x)$ versus x in Figure 1 that $f'(\overline{x}_1) < 0$; so that, $\overline{x}_1$ is asymptotically stable, by the Principle of Linearized Stability (PLS); and $f'(\overline{x}_2) > 0$; so that, $\overline{x}_2$) is unstable by the PLS.

   Possible solutions of (1) have been sketched in Figure 2. We see in Figure 2 that, if $x(0) < 5$, then

$$\lim_{t\to\infty} x(t) = 3.$$

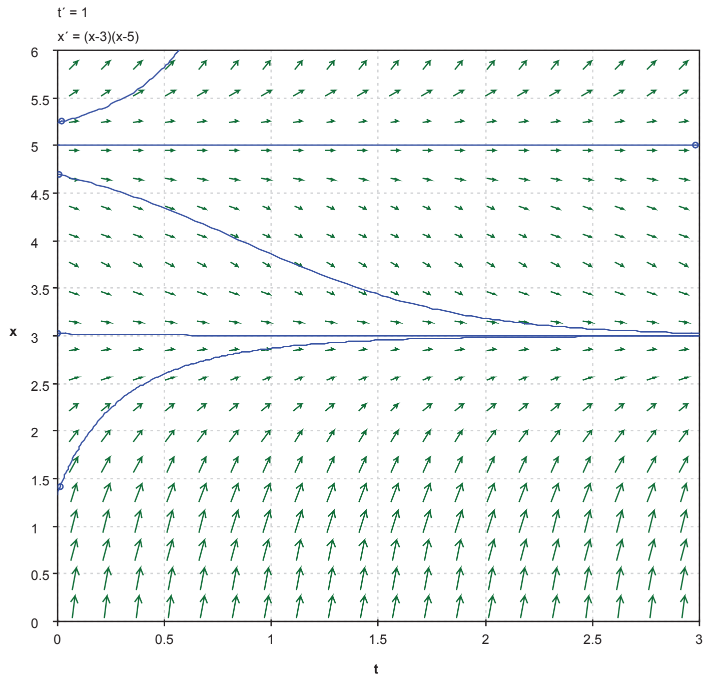

Figure 2: Possible Solutions of (1)

On the other hand, if $x(0) > 5$, the $x(t)$ increases without bound as t increases. $\square$

(b) $\dfrac{dx}{dt} = (1-x)(x-2)^2$
***Solution**:* Put

$$f(x) = (1-x)(x-2)^2, \quad \text{for } x \in \mathbb{R}.$$

We analyze the ODE

$$\frac{dx}{dt} = f(x). \tag{2}$$

The graph of $f(x)$ versus x is sketched in Figure 3. We see from the sketch in Figure 3 that the ODE in (2) has equilibrium solutions at

$$\overline{x}_1 = 1 \quad \text{and} \quad \overline{x}_2 = 2.$$

We also see from the sketch that $f'(\overline{x}_1) < 0$; so that, $\overline{x}_1 = 1$ is asymptotically stable, by the PLS. On the other hand, since $f'(\overline{x}_2) = 0$, the PLS does not apply in this case.

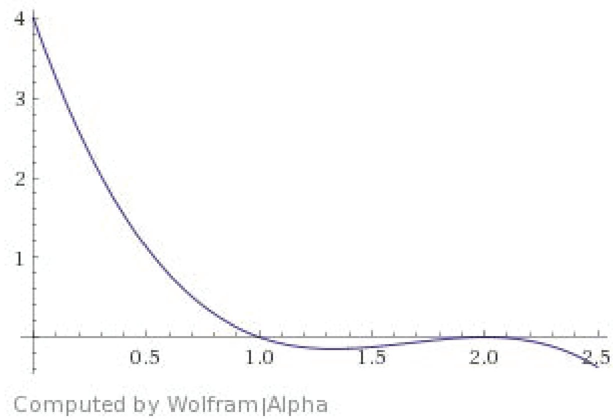

Figure 3: Sketch of $f(x)$ versus x

To determine the stability or non–stability of $\overline{x}_2$, observe that $f(x) < 0$ for $1 < x < 2$; so that, if $x(0) < 2$, but very close to 2, $x(t)$ will decrease, according to (2). Hence, $x(t)$ will tend away from 2 as t increases. Therefore, $\overline{x}_2 = 2$ is unstable.

Sketches of possible solutions on (2) are shown in Figure 4. From the sketches of possible solutions in Figure 4, we see that, if $x(0) \geqslant 2$, the

$$\lim_{t\to\infty} x(t) = 2.$$

On the other hand, is $x(0) < 2$, then

$$\lim_{t\to\infty} x(t) = 1.$$

□

2. The following equation models the evolution of a population that is being harvested at a constant rate:

$$\frac{dN}{dt} = 2N - 0.01N^2 - 75.$$

Find equilibrium solutions and sketch a few possible solution curves. According to model, what will happen if at time $t = 0$ the initial population densities are 40, 60, 150, or 170.

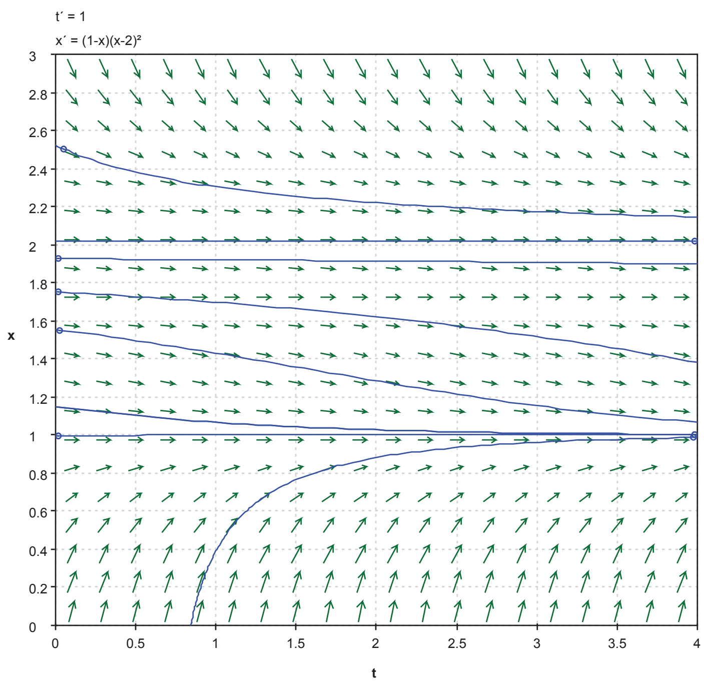

Figure 4: Possible Solutions of (2)

***Solution**:* Put

$$f(N) = 2N - 0.01N^2 - 75. \tag{3}$$

We would like to analyze the ODE

$$\frac{dN}{dt} = f(N). \tag{4}$$

Observe that the quadratic polynomial in (3) can be factored into

$$f(N) = -0.01(N - 50)(N - 150).$$

Thus, the ODE in (4) has two equilibrium solutions at

$$\overline{N}_1 = 50 \quad \text{and} \quad \overline{N}_2 = 150.$$

To determine the stability properties of these equilibrium solutions, consider the sketch of the graph of $f(N)$ versus N shown in Figure 5. We see from the sketch in Figure 5 that $f'(\overline{N}_1) > 0$; so that, $\overline{N}_1 = 50$ is unstable by the PLS. Similarly, we see that $f'(\overline{N}_2) < 0$; so that, $\overline{N}_2 = 150$ is asymptotically stable by the PLS.

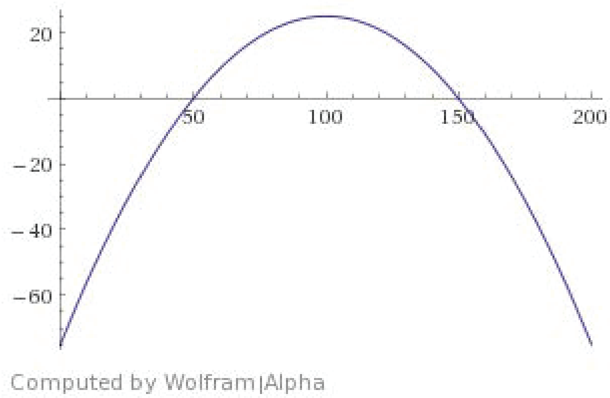

Figure 5: Sketch of graph of $f(N)$ versus N

Figure 6 shows sketches of possible solutions of the ODE in (4) obtained using `pplane`.

Examination of the sketches in Figure 6 leads to the following conclusions:

- If the initial population, $N(0)$, is less than 50, the population will die out in finite time.
- If $N(0) > 50$,
$$\lim_{t\to\infty} N(t) = 150.$$

□

3. For the following systems, sketch nullclines; find equilibrium points; apply the principle of Linearized stability (when applicable) to determine the stability properties of the equilibrium points; describe the local behavior trajectories near the equilibrium points; and sketch the phase portraits.

(a) $$\begin{cases} \dot{x} &= x^2 - y^2 - 1; \\ \dot{y} &= 2y. \end{cases}$$

***Solution**:* The $\dot{x} = 0$–nullcline is
$$x^2 - y^2 = 1 \quad \text{(hyperbola)},$$
and the $\dot{y} = 0$–nullcline is
$$y = 0 \quad \text{(the } x\text{–axis)}.$$

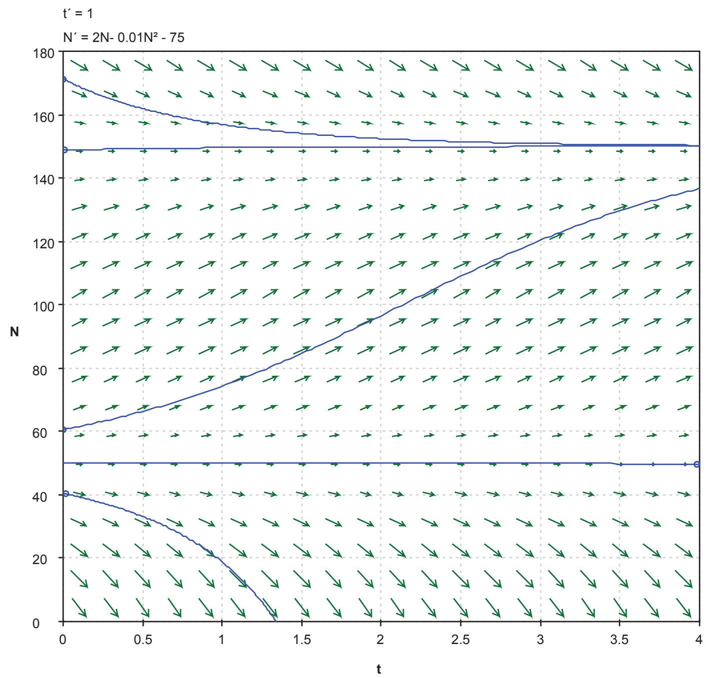

Figure 6: Possible Solutions of (4)

These are sketched in Figure 7. We see from the sketch in the figure that there are two equilibrium points:

$$(-1, 0) \quad \text{and} \quad (1, 0).$$

To determine the stability properties of the equilibrium points, we look at the derivative of the vector field

$$F(x, y) = \begin{pmatrix} x^2 - y^2 - 1 \\ 2y \end{pmatrix}, \qquad \text{for } (x, y) \in \mathbb{R}^2;$$

namely

$$DF(x, y) = \begin{pmatrix} 2x & -2y \\ 0 & 2 \end{pmatrix}, \qquad \text{for } (x, y) \in \mathbb{R}^2.$$

Computing

$$DF(-1, 0) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix},$$

we see that the eigenvalues of $DF(-1, 0)$ are -2 and 2; hence, by the PLS, $(-1, 0)$ is a saddle point.

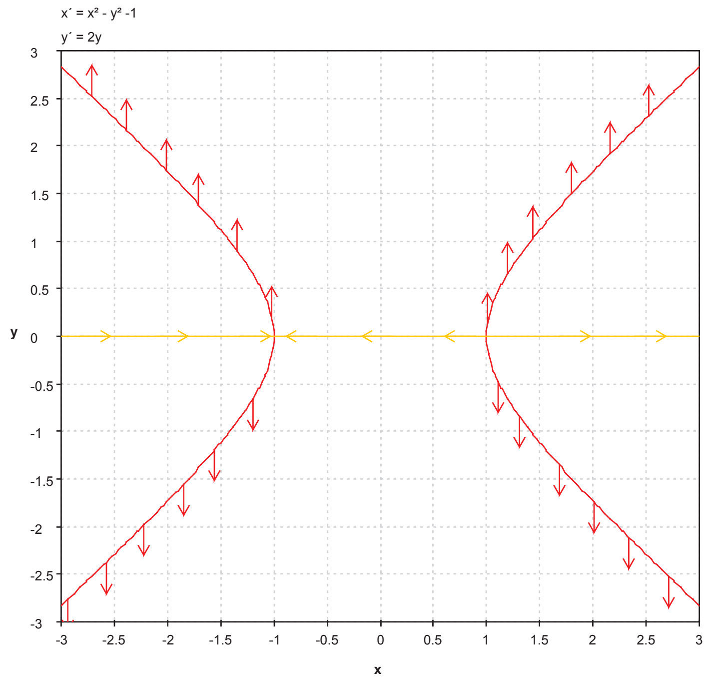

Figure 7: Sketch of Nullclines of the System in Problem 3(a)

Similarly, since

$$DF(1,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

the eigenvalues of $DF(1,0)$ are both positive; hence, $(1,0)$ is a source. A sketch of possible trajectories obtained using `pplane` is shown in Figure 8. □

(b) $\begin{cases} \dot{x} &= y - y^2 + 2; \\ \dot{y} &= 2x^2 - 2xy, \end{cases}$

Solution: The $\dot{x} = 0$–nullclines are the lines

$$y = -1 \quad \text{and} \quad y = 2,$$

and the $\dot{y} = 0$–nullclines are the lines

$$x = 0 \quad \text{(the } y\text{–axis)} \quad \text{and} \quad y = x.$$

These are sketched in Figure 9. We see from the sketch in the figure that

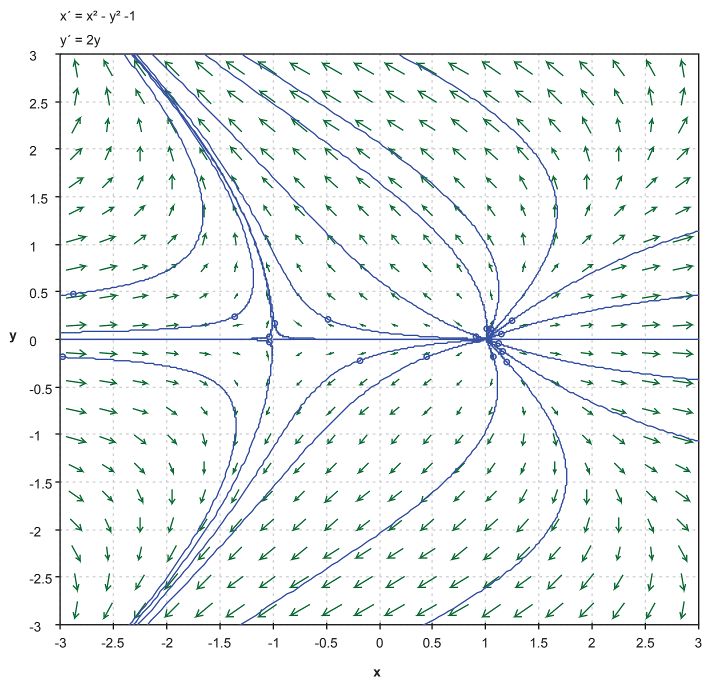

Figure 8: Sketch of Phase Portrait of the System in Problem 3(a)

there are four equilibrium points:

$$(-1,-1), \quad (0,-1), \quad (0,2) \quad \text{and} \quad (2,2).$$

To determine the stability properties of the equilibrium points, we look at the derivative of the vector field

$$F(x,y) = \begin{pmatrix} y - y^2 + 2 \\ 2x^2 - 2xy \end{pmatrix}, \quad \text{for } (x,y) \in \mathbb{R}^2;$$

namely

$$DF(x,y) = \begin{pmatrix} 0 & 1-2y \\ 4x-2y & -2x \end{pmatrix}, \quad \text{for } (x,y) \in \mathbb{R}^2.$$

Compute the characteristic polynomial of

$$DF(-1,-1) = \begin{pmatrix} 0 & 3 \\ -2 & 2 \end{pmatrix},$$

to get $p(\lambda) = \lambda^2 - 2\lambda + 6$. Thus, the eigenvalues of the linearization at $(-1,-1)$ are $1 \pm i\sqrt{5}$, which are complex with positive real part. Hence, by the PLS, $(-1,-1)$ is a spiral source.

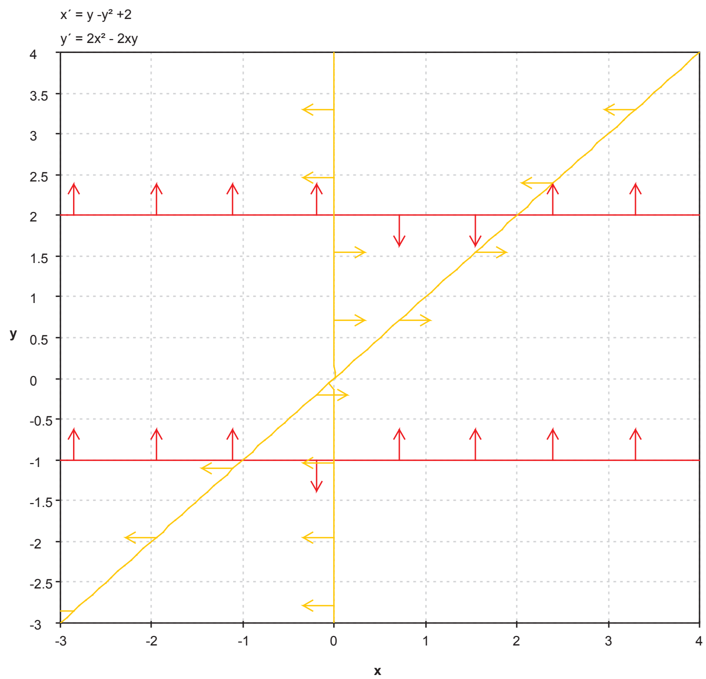

Figure 9: Sketch of Nullclines of the System in Problem 3(b)

Applying the PLS at the other equilibrium points, we get

$$DF(0,-1) = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix},$$

which has eigenvalues $\pm\sqrt{6}$, so that, $(1,0)$ is a saddle point;

$$DF(0,2) = \begin{pmatrix} 0 & -3 \\ -4 & 0 \end{pmatrix},$$

which has eigenvalues $\pm\sqrt{12}$, so that, $(0,2)$ is a saddle point;

$$DF(2,2) = \begin{pmatrix} 0 & -3 \\ 4 & -4 \end{pmatrix},$$

which has eigenvalues $-2 \pm 2\sqrt{2}$, which are complex with negative real part; so that, $(-2,2)$ is a s spiral sink.

A sketch of possible trajectories obtained using `pplane` is shown in Figure 10. □

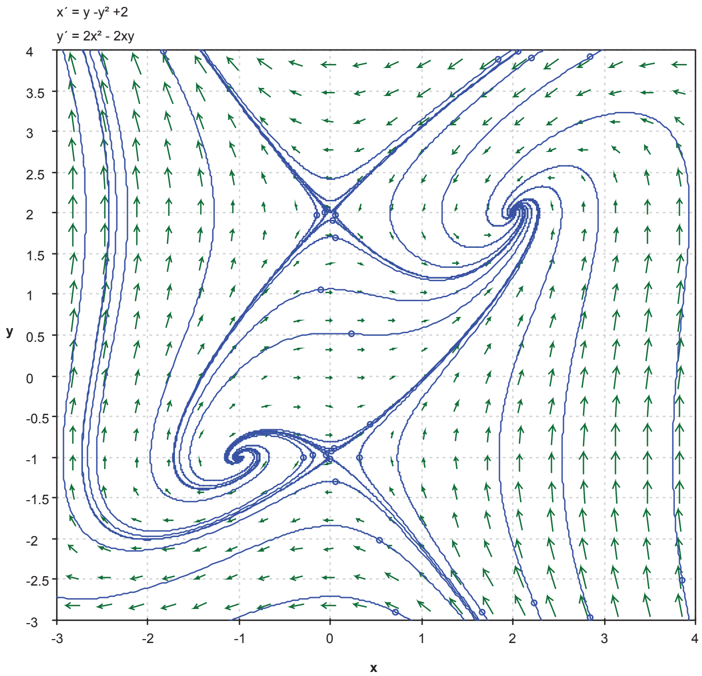

Figure 10: Sketch of Phase Portrait of the System in Problem 3(b)

(c) $\begin{cases} \dot{x} &= 4 - 2y; \\ \dot{y} &= 12 - 3x^2. \end{cases}$

***Solution**:* The $\dot{x} = 0$–nullcline is the line

$$y = 2,$$

and the $\dot{y} = 0$–nullclines are the lines

$$x = -2 \quad \text{and} \quad x = 2.$$

These are sketched in Figure 11. We see from the sketch in the figure that there are two equilibrium points:

$$(-2, 2) \quad \text{and} \quad (2, 2).$$

To determine the stability properties of the equilibrium points, we look at the derivative of the vector field

$$F(x, y) = \begin{pmatrix} 4 - 2y \\ 12 - 3x^2 \end{pmatrix}, \qquad \text{for } (x, y) \in \mathbb{R}^2;$$

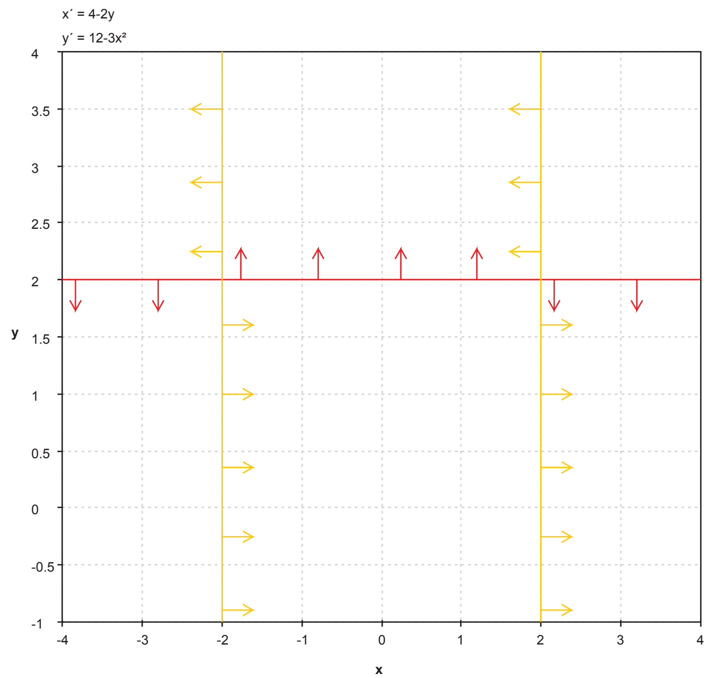

Figure 11: Sketch of Nullclines of the System in Problem 3(c)

namely

$$DF(x,y) = \begin{pmatrix} 0 & -2 \\ -6x & 0 \end{pmatrix}, \qquad \text{for } (x,y) \in \mathbb{R}^2.$$

Computing

$$DF(-2,2) = \begin{pmatrix} 0 & -2 \\ 12 & 0 \end{pmatrix},$$

we see that the eigenvalues of $DF(-2,2)$ are $\pm i2\sqrt{6}$, which are purely imaginary; hence, the PLS does not apply in this case.

Next, compute

$$DF(2,2) = \begin{pmatrix} 0 & -2 \\ -12 & 0 \end{pmatrix},$$

which a negative, $-2\sqrt{6}$, and a positive, $2\sqrt{6}$, eigenvalue. Thus, by the PLS, $(2,2)$ is a saddle point.

A sketch of possible trajectories obtained using `pplane` is shown in Figure 12. Observe that the sketch in Figure 12 suggests that $(-2,2)$ is a center for the system in Problem 3(c). □

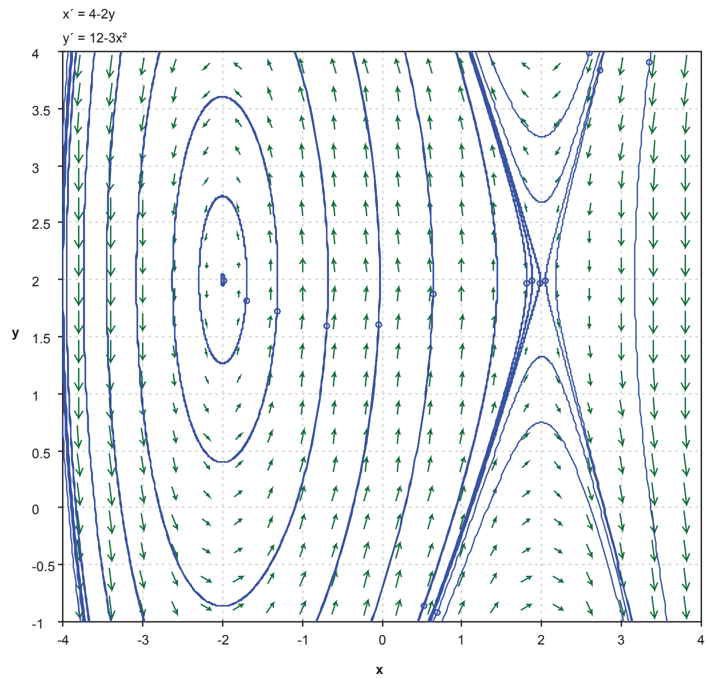

Figure 12: Sketch of Phase Portrait of the System in Problem 3(c)

(d) $\begin{cases} \dot{x} = x - x^3; \\ \dot{y} = -y, \end{cases}$

***Solution**:* The $\dot{x} = 0$–nullclines are the lines

$$x = -1, \quad x = 0 \quad \text{and} \quad x = 1,$$

and the $\dot{y} = 0$–nullcline is

$$y = 0 \quad (\text{the } x\text{–axis}).$$

These are sketched in Figure 13. We see from the sketch in the figure that there are three equilibrium points:

$$(-1, 0), \ (0, 0) \quad \text{and} \quad (1, 0).$$

To determine the stability properties of the equilibrium points, we look at the derivative of the vector field

$$F(x, y) = \begin{pmatrix} x - x^3 \\ -y \end{pmatrix}, \quad \text{for } (x, y) \in \mathbb{R}^2;$$

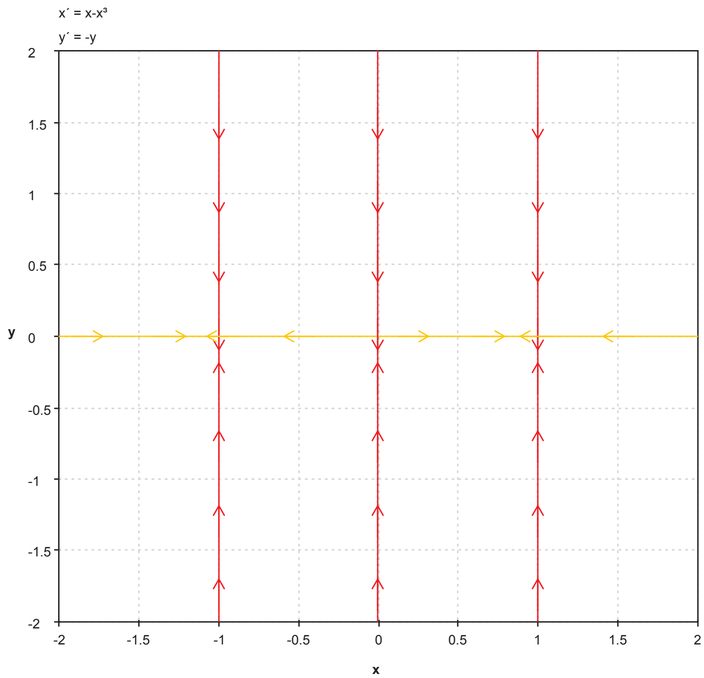

Figure 13: Sketch of Nullclines of the System in Problem 3(d)

namely

$$DF(x,y) = \begin{pmatrix} 1-3x^2 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \text{for } (x,y) \in \mathbb{R}^2.$$

Computing

$$DF(\pm 1, 0) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix},$$

which has negative eigenvalues -2 and -1; hence, by the PLS, $(-1,0)$ and (1,0) are sinks.

Similarly, since

$$DF(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

real eigenvalues of opposite signs, we conclude that $(0,0)$ is a saddle point, by the PLS.

A sketch of possible trajectories obtained using `pplane` is shown in Figure 14. □

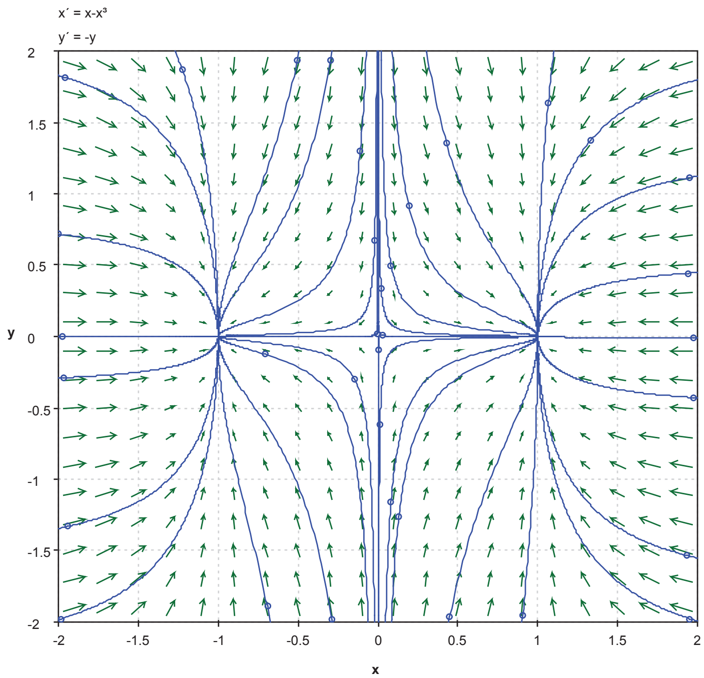

Figure 14: Sketch of Phase Portrait of the System in Problem 3(d)

4. **Gradient Systems**. Let F be a real–valued function with continuous partial derivatives defined in some domain, D, of $\mathbb{R}^2$. The system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \nabla F(x,y)$$

is called a gradient system.

Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $F(x,y) = x^2 - y^2$, for all $(x,y) \in \mathbb{R}^2$.

(a) Write down the gradient system associated with the function F.

***Solution**:* The gradient of F is

$$\nabla F(x,y) = \begin{pmatrix} 2x \\ -2y \end{pmatrix}, \quad \text{for all } (x,y) \in \mathbb{R}^2, \tag{5}$$

and the corresponding system is

$$\begin{cases} \dfrac{dx}{dt} = 2x; \\ \\ \dfrac{dy}{dt} = -2y. \end{cases} \tag{6}$$

□

(b) Find all equilibrium points of the system obtained in part (a) and determine the nature of their stability.

***Solution**:* The only equilibrium point of the system in (6) is the origin, and it is a saddle point. □

(c) Sketch the graph of the function F and sketch its level sets.

***Solution**:* A sketch of the graph of F in (5), obtained using WolframAlpha®, is shown in Figure 15. A sketch of the level sets of F, also obtained using

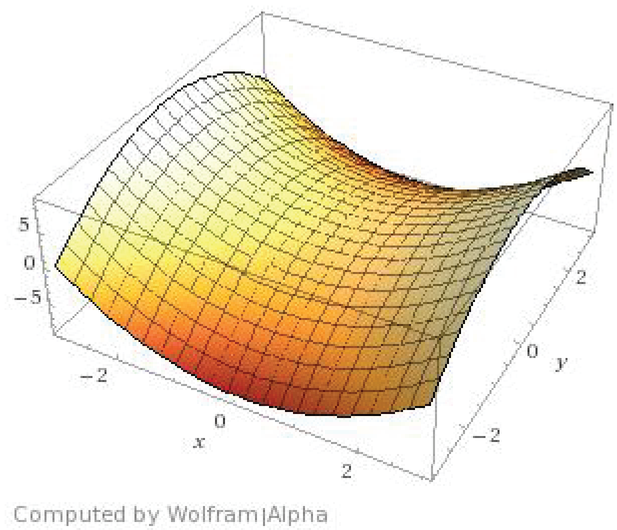

Figure 15: Sketch of Graph of F in (5)

WolframAlpha®, is shown in Figure 16. □

(d) Sketch the phase portrait of the system obtained in part (a).

***Solution**:* A sketch of the phase portrait of the system in (6) is shown in Figure 17. We note that the trajectories in the phase–portrait of the system in (6) are perpendicular to the level sets of F. This is to be expected be cause the gradient of a function if perpendicular to the level sets of the function. □

5. **Negative Gradient Flows.** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ denote a twice differentiable function with continuous partial derivatives. Consider the negative gradient

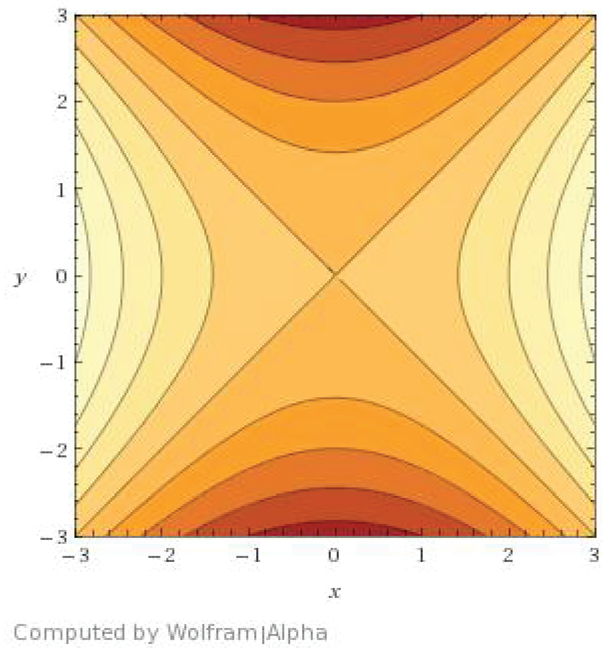

Figure 16: Sketch of Level Sets of F in (5)

system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -\nabla f(x, y). \tag{7}$$

(a) Let $(x(t), y(t))$ denote a solution curve of the system in (7) that contains no equilibrium points of (7). Show that f is strictly decreasing (with increasing t) along this trajectory.

***Solution**:* Rewrite the system in (7) as

$$\begin{cases} \dfrac{dx}{dt} = -\dfrac{\partial f}{\partial x}(x, y); \\ \\ \dfrac{dy}{dt} = -\dfrac{\partial f}{\partial y}(x, y), \end{cases} \tag{8}$$

and let $(x(t), y(t))$ denote a trajectory of the system that contains no equilibrium points of (7).

We consider the values of f on the trajectory $(x(t), y(t))$; namely, $f(x(t), y(t))$.

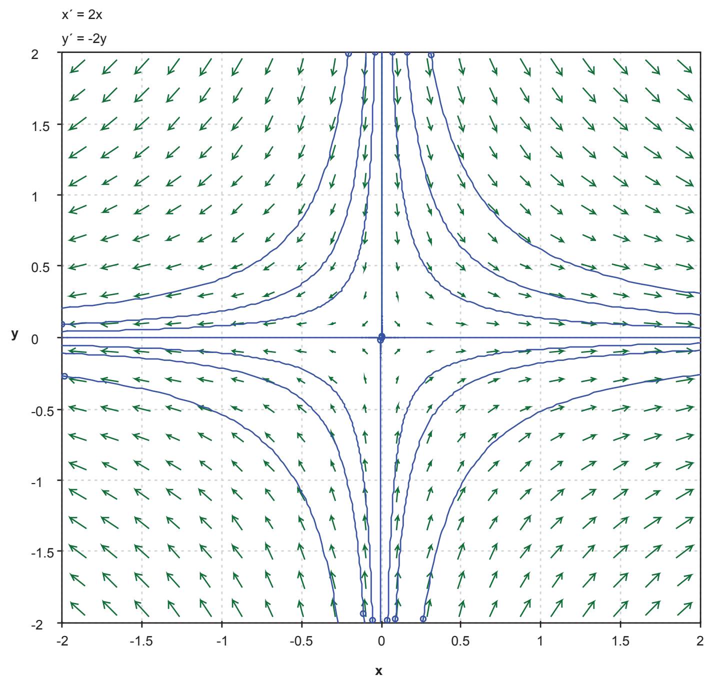

Figure 17: Sketch of Phase Portrait of System in (6)

Use the Chain Rule to compute

$$\frac{d}{dt}[f(x(t), y(t))] = \frac{\partial f}{\partial x}(x,y)\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}(x,y)\cdot\frac{dy}{dt};$$

so that, using the equations in (8),

$$\begin{aligned}\frac{d}{dt}[f(x(t), y(t))] &= -\frac{\partial f}{\partial x}(x,y)\cdot\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial y}(x,y)\cdot\frac{\partial f}{\partial y}(x,y)\\ &= -\left[\left(\frac{\partial f}{\partial x}(x,y)\right)^2 + \left(\frac{\partial f}{\partial y}(x,y)\right)^2\right],\end{aligned}$$

or

$$\frac{d}{dt}[f(x(t), y(t))] = -\|\nabla f(x,y)\|^2. \tag{9}$$

It follows from (9) and the assumption that $(x(t), y(t))$ contains no equilibrium points of the system (7) that

$$\frac{d}{dt}[f(x(t), y(t))] < 0, \quad \text{for all } t.$$

Hence, f is strictly decreasing (with increasing t) along the trajectory $(x(t), y(t))$. □

(b) Let $(x(t), y(t))$ denote a solution curve of the system in (7) that contains no equilibrium points of (7). Explain why this trajectory cannot be a cycle.

Solution: Suppose, by way of contradiction, that a trajectory, $(x(t), y(t))$ of the system in (7) that contains no equilibrium points of (7) is also a cycle. There there exists $T > 0$ such that

$$(x(T), y(T)) = (x(0), y(0)).$$

Then,

$$f(x(T), y(T)) = f(x(0), y(0)).$$

However, this contradicts the result from part (a), which says that

$$f(x(T), y(T)) < f(x(0), y(0)),$$

since f is strictly decreasing on the trajectory. This contradiction shows that this trajectory cannot be a cycle. □

6. **The Linear Pendulum Equation.** The pendulum equation (without friction),

$$\ell\ddot{\theta} = -g\sin(\theta), \tag{10}$$

can be linearized about the equilibrium position $\overline{\theta} = 0$ to yield the linear equation

$$\ell\ddot{\theta} = -g\theta. \tag{11}$$

The equation in (11) is the linearization of the equation in (10) and corresponds to oscillations of very small amplitude.

(a) Nondimsionalize the equation in (11) by introducing a dimensionless variable

$$\tau = \frac{t}{\lambda}. \tag{12}$$

What is the value of the parameter λ in terms of ℓ and g?

Solution: Use the Chain Rule to compute

$$\frac{d\theta}{d\tau} = \frac{d\theta}{dt}\cdot\frac{dt}{d\tau},$$

where, according to (12),

$$\frac{dt}{d\tau} = \lambda;$$

so that,

$$\frac{d\theta}{d\tau} = \lambda\frac{d\theta}{dt}.$$

Differentiating one more time and applying the Chain Rule again,

$$\frac{d^2\theta}{d\tau^2} = \lambda^2\frac{d^2\theta}{dt^2}.$$

Thus, using (11),

$$\frac{d^2\theta}{d\tau^2} = -\frac{\lambda^2 g}{\ell}\theta.$$

Setting

$$\frac{\lambda^2 g}{\ell} = 1, \tag{13}$$

we then get that

$$\frac{d^2\theta}{d\tau^2} = -\theta, \tag{14}$$

where, according to (13),

$$\lambda = \sqrt{\frac{\ell}{g}}. \tag{15}$$

□

(b) Solve the equation obtained in part (a) by first performing a phase–plane analysis.

***Solution**:* Turn the equation in (14) into a two–dimensional system, by setting

$$x = \theta \tag{16}$$

and

$$y = \frac{d\theta}{d\tau},$$

to get the system

$$\begin{cases} \dfrac{dx}{d\tau} = y \\[2ex] \dfrac{dy}{d\tau} = -x, \end{cases} \tag{17}$$

by virtue of (14).

The system in (17) is a linear system that can be solved by means of the fundamental matrix

$$\begin{pmatrix} \cos(\tau) & \sin(\tau) \\ -\sin(\tau) & \cos(\tau) \end{pmatrix}, \quad \text{for } \tau \in \mathbb{R},$$

to yield the general solution

$$\begin{pmatrix} x(\tau) \\ y(\tau) \end{pmatrix} = \begin{pmatrix} \cos(\tau) & \sin(\tau) \\ -\sin(\tau) & \cos(\tau) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad \text{for } \tau \in \mathbb{R},$$

and arbitrary constants c_1 and c_2, from which we obtain that

$$x(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau), \quad \text{for } \tau \in \mathbb{R};$$

which, in view of (16), yields the general solution of the ODE in (14):

$$\theta(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau), \quad \text{for } \tau \in \mathbb{R}. \tag{18}$$

Next, writing τ in terms of t, according to (12) and (13), we can express θ in (18) in terms of t to obtain the general solution

$$\theta(t) = c_1 \cos\left(\frac{t}{\sqrt{\ell/g}}\right) + c_2 \sin\left(\frac{t}{\sqrt{\ell/g}}\right), \quad \text{for } t \in \mathbb{R}. \tag{19}$$

□

(c) Compute the period, T, of oscillations of solutions of (11) in terms of ℓ and g.

***Solution**:* Note that the function defined in (19) is periodic of period T given by the expression

$$\frac{T}{\sqrt{\ell/g}} = 2\pi,$$

from which we get that

$$T = 2\pi\sqrt{\frac{\ell}{g}}.$$

□

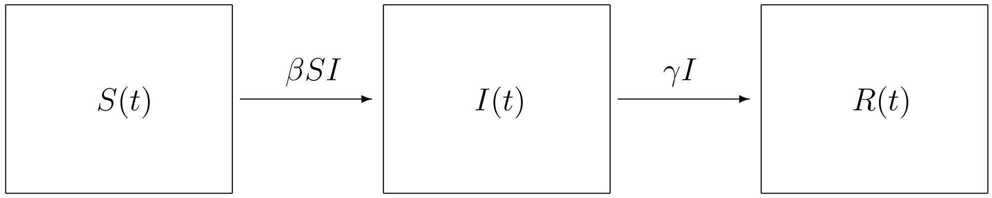

Figure 18: SIR Compartments

7. **Modeling the Spread of a Disease.** In a simple model for a disease that is spread through infections transmitted between individuals in a population, the population is divided into three compartments pictured in Figure 18. In the first compartment, $S(t)$ denotes the number of individuals in a population that are susceptible to acquiring the disease; in the second compartment, $I(t)$ denotes the number of infected individual who can also infect others; and, in the third compartment, $R(t)$ denotes the number of individuals who had the disease and who have recovered from it; they can no longer get infected.

   The arrows between compartments indicate the rates at which individuals flow from one compartment to the other. For instance, the arrow between the first two compartments indicates the transmission rate of the disease; it is assumed that the rate at which susceptible individuals get infected is proportional to product of number of susceptible individuals and the number of infected individuals with constant of proportionality $\beta > 0$. The rate at which infected individuals recover is indicated by the arrow between the last two compartments; it is assumed that this rate is proportional to the number of infected individuals, with constant of proportionality $\gamma > 0$.

   (a) What are the units for β and γ?

   Answer:
   β has units of $\dfrac{1}{\text{poppulation} \times \text{time}}$.
   γ has units of $\dfrac{1}{\text{time}}$. □

   (b) Use conservation principles to derive a system of differential equations for the functions S, I and R, assuming that they are differentiable, of the form

$$\begin{cases} \dfrac{dS}{dt} & = & f(S, I, R, \beta, \gamma); \\ \\ \dfrac{dI}{dt} & = & g(S, I, R, \beta, \gamma); \\ \\ \dfrac{dR}{dt} & = & h(S, I, R, \beta, \gamma), \end{cases} \tag{20}$$

where f, g and h are continuous functions that have continuous partial derivatives with respect to S, I and R. The system in (20) is known in the literature as the Kermack–McKendrick SIR model. It first appeared in the scientific literature in 1927.

***Solution**:* Use the conservation principle

$$\frac{dS}{dt} = \text{Rate of } S \text{ in} - \text{Rate of } S \text{ out},$$

where, according to the flow diagram in Figure 18,

$$\text{Rate of } S \text{ in} = 0$$

and

$$\text{Rate of } S \text{ out} = \beta S I;$$

so that

$$\frac{dS}{dt} = -\beta S I.$$

Similarly, we obtain

$$\frac{dI}{dt} = \beta S I - \gamma I,$$

and

$$\frac{dR}{dt} = \gamma I.$$

We therefore obtain the three–dimensional system

$$\begin{cases} \dfrac{dS}{dt} & = & -\beta S I; \\ \\ \dfrac{dI}{dt} & = & \beta S I - \gamma I; \\ \\ \dfrac{dR}{dt} & = & \gamma I. \end{cases} \tag{21}$$

□

(c) Deduce that the system in (20) implies that the total number of individuals in the population,

$$N(t) = S(t) + I(t) + R(t), \tag{22}$$

remains constant. Denote $N(t)$ by N, where N is a constant, for all t.

***Solution**:* Differentiate with respect to t on both sides of (22) to obtain that

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt};$$

so that, adding up the expressions on the right–hand side of (21),

$$\frac{dN}{dt} = 0, \qquad \text{for all } t.$$

It then follows that $N(t)$ is constant. Set

$$N(t) = N, \qquad \text{for all } t, \tag{23}$$

where N is a constant. □

(d) Explain why the result of part (c) implies that the study of the system (20) reduces to the study of the two–dimensional system

$$\begin{cases} \dfrac{dS}{dt} = f(S, I, R, \beta, \gamma); \\ \\ \dfrac{dI}{dt} = g(S, I, R, \beta, \gamma). \end{cases} \tag{24}$$

***Solution**:* Combining (22) and (23) we get

$$S(t) + I(t) + R(t) = N, \qquad \text{for all } t,$$

from which we get that

$$R(t) = N - S(t) - I(t), \qquad \text{for all } t. \tag{25}$$

It follows from (25) that, if $S(t)$ and $I(t)$ are known, then we can determine $R(t)$. Thus, it suffices to consider the two–dimensional system

$$\begin{cases} \dfrac{dS}{dt} = -\beta S I; \\ \\ \dfrac{dI}{dt} = \beta S I - \gamma I. \end{cases} \tag{26}$$

□

(e) Introduce dimensionless variables

$$x = \frac{S}{N}, \quad y = \frac{I}{N}, \qquad \text{and} \quad \tau = \frac{t}{\lambda}, \tag{27}$$

for some scaling factor, λ, in units of time, in order to write the system (24) in dimensionless form.

***Solution**:* Using the dimensionless variables given in (27), we nondimensionalize the system in (26).

Use the Chain Rule to compute

$$\frac{dx}{d\tau} = \frac{dx}{dt} \cdot \frac{dt}{d\tau},$$

where, according to the right–most expression in (27),

$$\frac{dt}{d\tau} = \lambda.$$

Thus, using the left–most expression in (27),

$$\frac{dx}{d\tau} = \frac{\lambda}{N}\frac{dS}{dt}.$$

Hence, substituting the first equation in (26),

$$\frac{dx}{d\tau} = -\lambda\beta N xy, \tag{28}$$

where we have also used the middle expression in (27).

Similar calculations to those leading to (28) can be used to derive the equation

$$\frac{dy}{d\tau} = \lambda\beta N xy - \lambda\gamma y. \tag{29}$$

Combining (28) and (29) yields the system

$$\begin{cases} \dfrac{dx}{d\tau} = -\lambda\beta N xy; \\[2ex] \dfrac{dy}{d\tau} = \lambda\beta N xy - \lambda\gamma y. \end{cases} \tag{30}$$

Observe that, in view of the answers to part (a) of this problem, the groupings of parameters $\lambda\beta N$ and $\lambda\gamma$ are dimensionless. We will set

$$\lambda\gamma = 1;$$

so that

$$\lambda = \frac{1}{\gamma},$$

and

$$\alpha = \lambda\beta N;$$

so that

$$\alpha = \frac{\beta N}{\gamma}. \tag{31}$$

With this choices of parameters we can rewrite the system in (30) as

$$\begin{cases} \dfrac{dx}{d\tau} & = & -\alpha xy; \\ \\ \dfrac{dy}{d\tau} & = & \alpha xy - y. \end{cases} \tag{32}$$

□

(f) Analyze the system obtained in part (e). What does the model in (20) predict about the spread of the disease in terms of the initial conditions $S(0) = S_o$, $I(0) = I_o$, $R(0) = 0$, and the parameters β, γ and N? Under which conditions will the number of infected individuals increase (an epidemic outbreak), or decrease?

***Solution**:* The $\dot{x} = 0$–nullclines of the system in (32) are

$$x = 0 \text{ (the } y\text{–axis)} \quad \text{and} \quad y = 0 \text{ (the } x\text{–axis)}.$$

The $\dot{y} = 0$–nullclines are the lines

$$y = 0 \text{ (the } x\text{–axis)} \quad \text{and} \quad x = \frac{1}{\alpha}.$$

These are sketched in Figure 19 for the case $\alpha = 1$. □ Thus, all the points in the points on the x–axis are equilibrium points of the system in (32). Hence, the system (32) has no isolated equilibrium points; therefore, the principle of linearized stability does not apply to the system (32).

Figure 19 also shows a few trajectories of the system in (32) obtained using the Java version of `pplane`. Note that, if the initial proportion of susceptible individuals,

$$x_o = \frac{S_o}{N},$$

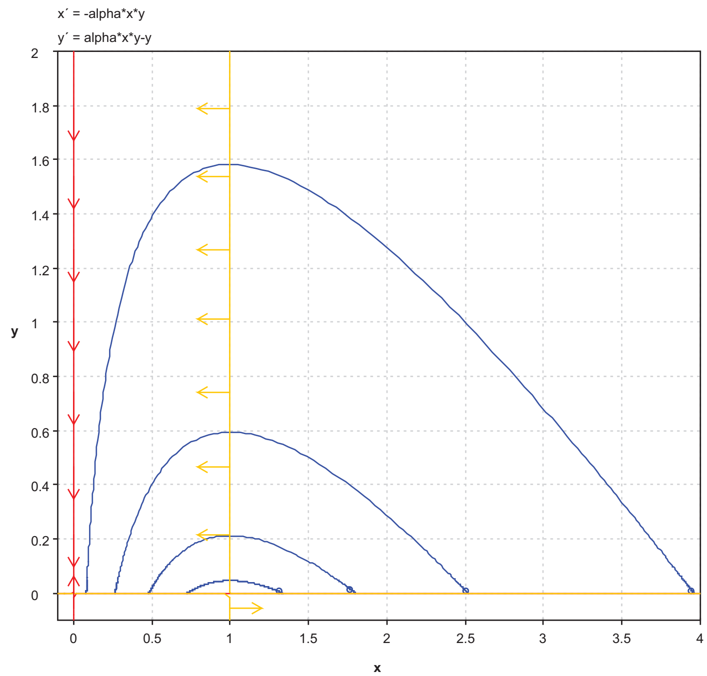

Figure 19: Sketch of Nullclines of the System in (32)

is bigger than $\dfrac{1}{\alpha}$, then the proportion of infected individuals, $\dfrac{I}{N}$, will increase. Thus, an epidemic outbreak will occur if

$$\frac{S_o}{N} > \frac{1}{\alpha},$$

or, in view of (31),

$$\frac{S_o}{N} > \frac{\gamma}{\beta N},$$

or

$$S_o > \frac{\gamma}{\beta}. \tag{33}$$

On the other hand, if

$$S_o \leqslant \frac{\gamma}{\beta},$$

the number of infected individuals will decrease to 0 in finite time. Hence, there will be not epidemic outbreak in this case.

Notice that the conclusion reached above is independent of the initial number of infected individuals, I_o. Thus, we can take $I_o = 1$; so that,

$S_o = N - 1$, in the case in which $R(0) = 0$. Thus, according to (33), there will be an epidemic outbreak if

$$N - 1 > \frac{\gamma}{\beta},$$

or

$$\frac{\beta(N-1)}{\gamma} > 1.$$

On the other hand, there will not be an outbreak if

$$\frac{\beta(N-1)}{\gamma} \leqslant 1.$$