Review Problems for Final Exam

1. The speed, v, of a falling skydiver is modeled by the differential equation

$$m\frac{dv}{dt} = mg - kv^2, \tag{1}$$

where m is the mass of the skydiver, g is the constant acceleration due to gravity near the surface of the earth, and k is the drag coefficient. Note that m, g and k are positive parameters.

(a) Give the units of the parameter k.

(b) Introduce dimensionless variables $u = \dfrac{v}{\mu}$ and $\tau = \dfrac{t}{\lambda}$ to write the equation in (1) in the dimensionless form

$$\frac{du}{d\tau} = f(u). \tag{2}$$

Express the scaling parameters μ and λ in terms of the original parameters m, g and k.

(c) Sketch the graph of f versus u, find the equilibrium points of the equation in (2), and use Principle of Linearized Stability (when applicable) to determine the nature of the stability of the equilibrium points.

(d) Sketch the shape of possible solution curves of the equation (2) in the τu–plane for various initial values.

(e) Use separation of variables and partial fractions to compute the general solution of the ODE in (2). Use this solution to obtain the general solution of the equation in (1).

(f) Use the solution of the equation in (1) obtained in the previous part to determine the terminal speed of the skydiver in terms of the original parameters m, g and k.

2. Consider the the nonlinear differential equation

$$\frac{du}{dt} = e^u - 1.$$

Find the equilibrium points of the equations and study their stability.

3. In this problem we show how small changes in the coefficients of system of linear equations can affect stability of an equilibrium point that is a center.

   (a) Consider the system $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Show that $(0,0)$ a center.

   (b) Next, consider $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \varepsilon & 1 \\ -1 & \varepsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, where $|\varepsilon| \neq 0$ is arbitrarily small. Show that no matter how small $|\varepsilon| \neq 0$ is, the center in part (a) becomes a spiral point. Discuss the stability–type for $\varepsilon > 0$ and for $\varepsilon < 0$.

4. Consider the second order, linear, homogeneous differential equation

$$\frac{d^2x}{dt^2} + \mu x = 0, \tag{3}$$

   where μ is a real parameter.

   (a) Give the general solution for each of the cases (i) $\mu < 0$, (ii) $\mu = 0$ and (iii) $\mu > 0$.

   (b) For each of the cases (i), (ii) and (iii) in part (a), determine conditions on μ (in any) that will guarantee that the equation in (3) has a nontrivial solution $x\colon \mathbb{R} \to \mathbb{R}$ satisfying $x(0) = 0$ and $x(\pi) = 0$.

5. Give the general solution of the system $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

6. The system of differential equations

$$\begin{cases} \dfrac{dx}{dt} &= x(2 - x - y); \\[2ex] \dfrac{dy}{dt} &= y(3 - 2x - y) \end{cases}$$

   describes competing species of densities $x \geqslant 0$ and $y \geqslant 0$. Explain why these equations make it mathematically possible, but extremely unlikely, for both species to coexist.

7. Consider the two–dimensional, autonomous system

$$\begin{cases} \dfrac{dx}{dt} & = & (x-y)(1-x^2-y^2); \\[2ex] \dfrac{dy}{dt} & = & (x+y)(1-x^2-y^2). \end{cases}$$

   (a) Verify that every point in the unit circle, $C = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$, is an equilibrium point.

   (b) Show that $(0,0)$ is an isolated equilibrium point of the system.

   (c) Determine the nature of the stability of $(0,0)$.

   (d) Let D denote the open unit disc in $\mathbb{R}^2$,

$$D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}.$$

   Show that every trajectory that starts at a point $(x_o, y_o) \in D$, such that $(x_o, y_o) \neq (0,0)$, will tend towards C as $t \to \infty$.

   (e) Show that every trajectory that starts at a point $(x_o, y_o) \in \mathbb{R}^2$, such that $x_o^2 + y_o^2 > 1$, will tend towards C as $t \to \infty$.

8. Consider the two–dimensional, autonomous system $\begin{cases} \dot{x} & = & y; \\ \dot{y} & = & 4x - x^3. \end{cases}$

   (a) Sketch nullclines, compute equilibrium points, and use the Principle of Linearized Stability (when applicable) to determine the nature of the stability of the equilibrium points.

   (b) Find a conserved quantity for the system.

   (c) Discuss the phase–portrait of the system.

9. Consider the two–dimensional, autonomous system

$$\begin{cases} \dot{x} & = & x - y - x(x^2+y^2); \\ \dot{y} & = & x + y - y(x^2+y^2). \end{cases}$$

   (a) Show that $(0,0)$ is an isolated equilibrium point of the system.

   (b) Determine the nature of the stability of $(0,0)$.

10. The system of differential equations

$$\begin{cases} \dfrac{dx}{dt} & = & \dfrac{c}{a+ky} - b; \\ \\ \dfrac{dy}{dt} & = & \gamma x - \beta, \end{cases}$$

models the time evolution of the interaction of an enzyme of concentration, y, and m–RNA, of concentration x, in a process of protein synthesis. The parameters a, b, c, k, α and β are assumed to be positive. This model was proposed by Brian C. Goodwin in 1965 (*Oscillatory behavior in enzymatic control processes,* in Advances in Enzyme Regulation, Volume 3, 1965, Pages 425–428, IN1–IN2, 429430, IN3–IN6, 431–437).

(a) Sketch the nullclines, find all equilibrium points, and apply the Principle of Linearized Stability (when applicable) to determine the nature of the stability of the equilibrium points.

(b) Find a conserved quantity for the the system.

(c) Discuss the phase–portrait of the system.