## Assignment #1

**Due on Wednesday, January 24, 2018**

**Read** Section 2.3.3 on the *Vibrating String,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

**Read** Section 2.4 on *Modeling Small Amplitude Vibrations,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

**Do** the following problems

1. In Sections 2.3.3 and 2.4 in the lecture notes at
`http://pages.pomona.edu/~ajr04747/`

   we derived the one–dimensional, linear, homogeneous wave equation

$$u_{tt} - c^2 u_{xx} = 0, \tag{1}$$

   where $c$ is a positive constant, for a function $u\colon [0, L] \times [0, \infty) \to \mathbb{R}$ of the variables $x \in [0, L]$ and $t \geqslant 0$ that has continuous second partial derivatives in $x \in (0, L)$ and $t > 0$; we write $u \in C^2((0, L) \times (0, \infty))$.

   (a) Assume that $u \in C^2((0, L) \times (0, \infty))$ solves the PDE in (1) and let

$$v(x, t) = au(x, t), \qquad \text{for all } x \in [0, L] \text{ and } t \geqslant 0,$$

   where $a$ is a real constant. Show that $v$ is also a solution on (1).

   (b) Let $u_1 \in C^2((0, L) \times (0, \infty))$ and $u_2 \in C^2((0, L) \times (0, \infty))$ denote two solutions of (1). Show that $u_1 + u_2$ is also a solution of (1).

2. For each of the following functions, $u \in C^2((0, L) \times (0, \infty))$, verify that $u$ solves the wave equation in (1).

   (a) $u(x, t) = x - ct$, for all $x \in \mathbb{R}$ and all $t \in \mathbb{R}$.

   (b) $u(x, t) = x^2 - 2cxt + c^2t^2$, for all $x \in \mathbb{R}$ and all $t \in \mathbb{R}$.

3. Define $u\colon \mathbb{R}^2 \to \mathbb{R}$ by $u(x, t) = x^2 + t^2$, for all $(x, t) \in \mathbb{R}^2$.

   Show that $u$ is not a solution of the PDE in (1), unless $c = \pm 1$.

4. Let $F\colon \mathbb{R} \to \mathbb{R}$ and $G\colon \mathbb{R} \to \mathbb{R}$ denote real valued functions of a single variable that are assumed to be twice differentiable.

   (a) Define $u_1(x,t) = F(x - ct)$ for all $x \in \mathbb{R}$ and all $t \in \mathbb{R}$. Verify that $u_1$ is a solution of the PDE in (1).

   (b) Define $u_2(x,t) = G(x + ct)$ for all $x \in \mathbb{R}$ and all $t \in \mathbb{R}$. Verify that $u_2$ is a solution of the PDE in (1).

   (c) Define $u(x,t) = u_1(x,t) + u_2(x,t)$ for all $(x,t) \in \mathbb{R}^2$, where $u_1$ and $u_2$ are as in parts (a) and (b) above, respectively. Verify that $u$ is a solution of the PDE in (1).

5. In Problem 4 you verified that the function

$$u(x,t) = F(x - ct) + G(x + ct), \quad \text{for } x \in \mathbb{R} \text{ and } t \in \mathbb{R}, \tag{2}$$

where $F\colon \mathbb{R} \to \mathbb{R}$ and $G\colon \mathbb{R} \to \mathbb{R}$ are twice–differentiable functions of a single real variable, is a solution of the PDE in (1). In this problem, we use the formula in (2) to find a formula for a solution of the vibrating string equation satisfying the boundary conditions

$$u(0,t) = 0, \quad \text{for all } t > 0, \tag{3}$$

and

$$u(L,t) = 0, \quad \text{for all } t > 0,$$

and the initial conditions

$$u(x,0) = f(x), \quad \text{for all } x \in [0,L], \tag{4}$$

for some function $f\colon \mathbb{R} \to \mathbb{R}$, and

$$u_t(x,0) = 0, \quad \text{for all } x \in [0,L], \tag{5}$$

   (a) Assume that the function $f$ in (4) is differentiable. Show that Use the initial conditions in (4) and (5) to show that $F$ and $G$ solve the system

$$\begin{cases} F + G = f \\ F' - G' = 0 \end{cases} \tag{6}$$

(b) Solve the system in (6) to show that

$$F(x) = \frac{1}{2}f(x) - C \tag{7}$$

and

$$G(x) = \frac{1}{2}f(x) + C, \tag{8}$$

for some constant of integration $C$.

(c) Use the expressions in (8), (7) and (2) to show that any solution of the wave equation in (1), which is of the form given in (2) and satisfies the initial conditions in (4) and (5), must be of the form

$$u(x,t) = \frac{1}{2}f(x-ct) + \frac{1}{2}f(x+ct), \qquad \text{for } x \in [0,L] \text{ and } t \geqslant 0. \tag{9}$$

(d) Use the boundary condition in (3) and the formula for $u$ in (9) to deduce that the function $f\colon \mathbb{R} \to \mathbb{R}$ must be an odd function.