## Assignment #2

**Due on Friday, February 2, 2018**

**Read** Section 5.1 on *Solving the Vibrating String Equation* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

**Read** Section 1.5.1 on *Separation of Variables* in Gustafson.

**Background and Definitions**
In these problem set we show that the following initial–boundary value problem for the one–dimensional wave equation has at most one solution:

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t), & \text{for } x \in (0,L) \text{ and } t > 0; \\ u(0,t) = a(t), & \text{for } t \geqslant 0; \\ u(L,t) = b(t), & \text{for } t \geqslant 0; \\ u(x,0) = f(x), & \text{for } x \in [0,L]; \\ u_t(x,0) = g(x), & \text{for } x \in [0,L], \end{cases} \tag{1}$$

where the function $F\colon [0,L]\times[0,+\infty) \to \mathbb{R}$ is assumed to be continuous; $f\colon [0,L] \to \mathbb{R}$ and $g\colon [0,L] \to \mathbb{R}$ are continuous on $[0,L]$; $a\colon [0,+\infty) \to \mathbb{R}$ and $b\colon [0,+\infty) \to \mathbb{R}$ are continuous functions; and $c$ is a positive constant.

We will use information about the following related homogeneous problem

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0, & \text{for } x \in (0,L) \text{ and } t > 0; \\ v(0,t) = 0, & \text{for } t \geqslant 0; \\ v(L,t) = 0, & \text{for } t \geqslant 0; \\ v(x,0) = 0, & \text{for } x \in [0,L]; \\ v_t(x,0) = 0, & \text{for } x \in [0,L]. \end{cases} \tag{2}$$

**Do** the following problems

1. Let $v \in C^2([0,L] \times [0,+\infty), \mathbb{R})$ and define

$$E(t) = \frac{1}{2}\int_0^L [c^2(v_x(x,t))^2 + (v_t(x,t))^2]\, dx, \qquad \text{for } t \geqslant 0. \tag{3}$$

   (a) Use the results in the Appendix on **Differentiating Under the Integral Sign** in the lecture notes at `http://pages.pomona.edu/~ajr04747/` to show that the function $E\colon [0,+\infty) \to \mathbb{R}$ defined in (3) is differentiable for all $t > 0$.

(b) Give a formula for computing $E'(t)$ for all $t > 0$.

2. Let $v \in C^2([0, L] \times [0, +\infty), \mathbb{R})$ satisfy the boundary conditions in the BVP in (2).

   (a) Show that
   $$v_t(0, t) = 0, \qquad \text{for } t > 0, \tag{4}$$
   and
   $$v_t(L, t) = 0, \qquad \text{for } t > 0. \tag{5}$$

   (b) Use integration by parts and the results in (4) and (5) to show that
   $$\int_0^L v_x(x, t) v_{tx}(x, t)\ dx = -\int_0^L v_t(x, t) v_{xx}(x, t)\ dx, \qquad \text{for all } t > 0. \tag{6}$$

3. Use the results in part (b) of Problem 1 and part (b) of Problem 2 to show that, if $v \in C^2([0, L] \times [0, +\infty), \mathbb{R})$ satisfies the boundary conditions in the BVP in (2), then
   $$\frac{dE}{dt} = \int_0^L v_t(x, t)[v_{tt}(x, t) - c^2 v_{xx}(x, t)]\ dx, \qquad \text{for all } t > 0. \tag{7}$$

4. Let $v \in C^2([0, L] \times [0, +\infty), \mathbb{R})$ be a solution of the homogeneous BVP (2) and let $E\colon [0, +\infty) \to \mathbb{R}$ be as defined in (3) in Problem 1.

   (a) Use the result from Problem 3 to deduce that $E(t) = 0$ for all $t \geqslant 0$.

   (b) Deduce that $v(x, t) = 0$ for all $x \in [0, L]$ and all $t \geqslant 0$.

5. Use the result of part (b) of Problem 4 to prove that the initial–boundary value problem (1) can have at most one solution.

   *Suggestion:* Assume that
   $$u_1 \in C^2([0, L] \times [0, +\infty), \mathbb{R}) \text{ and } u_2 \in C^2([0, L] \times [0, +\infty), \mathbb{R})$$
   are solutions of the BVP (1) and define
   $$v(x, t) = u_1(x, t) - u_2(x, t), \qquad \text{for } x \in [0, L] \text{ and } t \geqslant 0.$$