Assignment #3

Due on Friday, February 9, 2018

Read Section 5.1.2 on *Fourier Series Expansions* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Background and Definitions
Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a bounded, periodic function of period $2L$, where $L > 0$. Assume also that f is integrable over $[-L, L]$; so that,

$$\int_{-L}^{L} |f(x)|\ dx < \infty, \tag{1}$$

where the integral in (1) denotes the Riemann integral.

Fourier Coefficients of f. The Fourier coefficients of f are defined to be

$$a_o = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx; \tag{2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx, \qquad \text{for } n = 1, 2, 3, \ldots; \tag{3}$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx, \qquad \text{for } n = 1, 2, 3, \ldots. \tag{4}$$

Fourier Series Expansion of f. The Fourier series expansion of f is the trigonometric series

$$a_o + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right], \qquad \text{for } x \in [-L, L]. \tag{5}$$

Do the following problems

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bounded, $2L$–periodic function that is integrable over $[-L, L]$.

   (a) Show that, for any real number r,

   $$\int_{r-L}^{r+L} f(x)\ dx = \int_{-L}^{L} f(x)\ dx.$$

   Deduce therefore that, for any real numbers c and d such that $d > c$ and $d - c = 2L$,

   $$\int_{c}^{d} f(x)\ dx = \int_{-L}^{L} f(x)\ dx.$$

(b) Show that the Fourier coefficients of f are also given by

$$a_o = \frac{1}{2L}\int_0^{2L} f(x)\ dx;$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx, \quad \text{for } n = 1, 2, 3, \ldots;$$

and

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx, \quad \text{for } n = 1, 2, 3, \ldots$$

2. Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is bounded, $2L$–periodic, and integrable over $[-L, L]$. Assume also that f is even.

(a) Show that the Fourier coefficients of f are given by

$$a_o = \frac{1}{L}\int_0^{L} f(x)\ dx;$$

$$a_n = \frac{2}{L}\int_0^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx, \quad \text{for } n = 1, 2, 3, \ldots;$$

and

$$b_n = 0, \quad \text{for } n = 1, 2, 3, \ldots$$

(b) Give the Fourier series expansion for f in this case.

3. Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is bounded, $2L$–periodic, and integrable over $[-L, L]$. Assume also that f is odd.

(a) Show that the Fourier coefficients of f are given by

$$a_o = 0, \quad \text{for } n = 0, 1, 2, 3, \ldots$$

and

$$b_n = \frac{2}{L}\int_0^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx, \quad \text{for } n = 1, 2, 3, \ldots.$$

(b) Give the Fourier series expansion for f in this case.

4. **The Riemann–Lebesgue Lemma**. Assume that $f\colon \mathbb{R}\to\mathbb{R}$ is bounded and $2L$–periodic. Assume also that f is square integrable over $[-L, L]$; that is,

$$\int_{-L}^{L} |f(x)|^2\, dx < \infty.$$

   (a) Use the Cauchy–Schwarz inequality to show that f is also integrable over $[-L, L]$.

   (b) Let a_n, for $n = 0, 1, 2, 3, \ldots$, and b_n, for $n = 1, 2, 3, \ldots$, denote the Fourier coefficients of f. The Riemann–Lebesgue Lemma states that

$$\lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \lim_{n\to\infty} b_n = 0.$$

   Use integration by parts to prove the Riemann–Lebesgue Lemma for the special case in which $f \in C^1(\mathbb{R}, \mathbb{R})$.

5. Let $f\colon [0, L] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{2x}{L}, & \text{if } 0 \leqslant x \leqslant \dfrac{L}{2}; \\[2ex] \dfrac{2}{L}(L - x), & \text{if } \dfrac{L}{2} < x \leqslant L. \end{cases}$$

   (a) Sketch the odd, periodic extension of f over the interval $[-2L, 2L]$.

   (b) Compute the Fourier coefficients of f.

   (c) Give the Fourier series expansion for f.

   (d) Use graphing software to sketch the approximations

$$S_N(x) = \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi x}{L}\right), \qquad \text{for } x \in [-L, L],$$

   for N equal to 1, 3 and 5. You may take $L = \pi$ in your sketch.