Assignment #5

Due on Friday, March 9, 2018

Read Section 5.3 on *The Dirichlet Problem for the Unit Disc* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 1.6.1 on *Divergence Theorem*, pp. 46–57, in *Introduction to Partial Differential Equations and Hilbert Space Methods* by Karl E. Gustafson.

Background and Definitions

Divergence. Let $\mathcal{U}$ be an open subset of $\mathbb{R}^2$ and $\overrightarrow{F} \in C^1(\mathcal{U}, \mathbb{R}^2)$ be a vector field given by

$$\overrightarrow{F}(x,y) = (P(x,y), Q(x,y)), \quad \text{for } (x,y) \in \mathcal{U},$$

where $P \in C^1(\mathcal{U}, \mathbb{R})$ and $Q \in C^1(\mathcal{U}, \mathbb{R})$ are C^1, real–valued functions defined on $\mathcal{U}$. The divergence of $\overrightarrow{F}$, denoted $\operatorname{div}\overrightarrow{F}$, is the scalar field, $\operatorname{div}\overrightarrow{F} : \mathcal{U} \to \mathbb{R}$ defined by

$$\operatorname{div}\overrightarrow{F}(x,y) = \frac{\partial P}{\partial x}(x,y) + \frac{\partial Q}{\partial y}(x,y)), \quad \text{for } (x,y) \in \mathcal{U}.$$

Gradient. Let $\mathcal{U}$ be an open subset of $\mathbb{R}^2$ and $u \in C^1(\mathcal{U}, \mathbb{R})$ be a scalar field. The gradient of u, denoted ∇u, is the vector field, $\nabla u \colon \mathcal{U} \to \mathbb{R}^2$ defined by

$$\nabla u(x,y) = \left(\frac{\partial u}{\partial x}(x,y), \frac{\partial u}{\partial y}(x,y) \right), \quad \text{for } (x,y) \in \mathcal{U}.$$

Laplacian. Let $\mathcal{U}$ be an open subset of $\mathbb{R}^2$ and $u \in C^2(\mathcal{U}, \mathbb{R})$ be a scalar field. The divergence of the gradient of u, $\operatorname{div}\nabla u$, is called the Laplacian of u, denoted by Δu. Thus,

$$\Delta u = \operatorname{div}\nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

or

$$\Delta u = u_{xx} + u_{yy}.$$

Laplace's Equation and Harmonic Functions. Let $\mathcal{U}$ denote an open subset of $\mathbb{R}^2$. A function $u \in C^2(\mathcal{U}, \mathbb{R})$ is said to satisfy Laplace's equation in $\mathcal{U}$ if

$$u_{xx} + u_{yy} = 0 \quad \text{in } \mathcal{U}. \tag{1}$$

A function $u \in C^2(\mathcal{U}, \mathbb{R})$ satisfying the PDE in (1) is said to be **harmonic** in $\mathcal{U}$.

The Divergence Theorem in $\mathbb{R}^2$. Let $\mathcal{U}$ be an open subset of $\mathbb{R}^2$ and Ω an open subset of $\mathcal{U}$ such that $\overline{\Omega} \subset \mathcal{U}$. Suppose that Ω is bounded with boundary $\partial\Omega$. Assume that $\partial\Omega$ is a piecewise C^1, simple, closed curve. Let $\overrightarrow{F} \in C^1(\mathcal{U}, \mathbb{R}^2)$. Then,

$$\iint_{\Omega} \operatorname{div}\overrightarrow{F}\ dxdy = \oint_{\partial\Omega} \overrightarrow{F} \cdot \widehat{n}\ ds, \tag{2}$$

where $\widehat{n}$ is the outward, unit, normal vector to $\partial\Omega$ that exists everywhere on $\partial\Omega$, except possibly at finitely many points.

Do the following problems.

1. Let $\mathcal{U}$ be an open subset of $\mathbb{R}^2$, $\overrightarrow{F} \in C^1(\mathcal{U}, \mathbb{R}^2)$ be a vector field and $u, v \in C^1(\mathcal{U}, \mathbb{R})$ be a scalar fields.

   (a) Derive the identity: $\operatorname{div}(u\overrightarrow{F}) = \nabla u \cdot \overrightarrow{F} + u \operatorname{div}\overrightarrow{F}$, where $\nabla u \cdot \overrightarrow{F}$ denotes the dot–product of ∇u and $\overrightarrow{F}$.

   (b) Derive the identity: $\operatorname{div}(v\nabla u) = \nabla v \cdot \nabla u + v\ \Delta u$, where $\nabla v \cdot \nabla u$ denotes the dot–product of ∇v and ∇u, and Δu is the Laplacian of u.

2. Let $\mathcal{U}$ be an open subset of $\mathbb{R}^2$ and Ω be an open subset of $\mathbb{R}^2$ such that $\overline{\Omega} \subset \mathcal{U}$. Assume that the boundary, $\partial\Omega$, of Ω is a simple closed curve parametrized by $\sigma \in C^1([0,1], \mathbb{R}^2)$. Let $u \in C^2(\mathcal{U}, \mathbb{R})$ and $v \in C^1(\mathcal{U}, \mathbb{R})$. Apply the Divergence Theorem (2) to the vector field $\overrightarrow{F} = v\nabla u$ to obtain

$$\iint_{\Omega} \nabla u \cdot \nabla v\ dxdy + \iint_{\Omega} v\Delta u\ dxdy = \oint_{\partial\Omega} v\frac{\partial u}{\partial n}\ ds, \tag{3}$$

   where Δu is the Laplacian of u and $\dfrac{\partial u}{\partial n}$ is the directional derivative of u in the direction of a unit vector perpendicular to $\partial\Omega$ which points away from Ω. This is usually referred to as **Green's identity I** (see p. 47 in Gustafson's book).

3. Let $\mathcal{U}$ be an open subset of $\mathbb{R}^2$ and Ω be an open subset of $\mathbb{R}^2$ such that $\overline{\Omega} \subset \mathcal{U}$. Assume that the boundary, $\partial\Omega$, of Ω is a simple closed curve parametrized by $\sigma \in C^1([0,1], \mathbb{R}^2)$. Put $C^1_o(\Omega, \mathbb{R}) = \{v \in C^1(\mathcal{U}, \mathbb{R}) \mid v = 0 \text{ on } \partial\Omega\}$; that is, $C^1_o(\Omega, \mathbb{R})$ is the space of C^1 functions in Ω that vanish on the boundary of Ω. Let $u \in C^2(\mathcal{U}, \mathbb{R})$. Use Green's identity I in (3) to show that

$$\iint_{\Omega} \nabla v \cdot \nabla u\ dxdy = -\iint_{\Omega} v\Delta u\ dxdy, \qquad \text{for all } v \in C^1_o(\Omega, \mathbb{R}).$$

4. Let $\mathcal{U}$ and Ω be as in Problem 3.

   (a) Use the result from Problem 3 to show that, for any $u \in C^2(\mathcal{U}, \mathbb{R})$ that is harmonic in Ω,

   $$\iint_\Omega \nabla u \cdot \nabla v \; dxdy = 0, \qquad \text{for all } v \in C_o^1(\Omega, \mathbb{R}).$$

   (b) Assume that $u \in C^2(\mathcal{U}, \mathbb{R})$ is harmonic in Ω. Show that, if $u = 0$ on $\partial\Omega$, then $u(x, y) = 0$ for all $(x, y) \in \Omega$.

5. Let $\mathcal{U}$ be an open subset of $\mathbb{R}^2$ and Ω be an open subset of $\mathbb{R}^2$ such that $\overline{\Omega} \subset \mathcal{U}$. Assume that the boundary, $\partial\Omega$, of Ω is piecewise C^1.

   Let $f \in C(\mathcal{U}, \mathbb{R})$ and $g \in C(\mathcal{U}, \mathbb{R})$ be given functions. Use the result of Problem 4 to show that the boundary value problem

   $$\begin{cases} u_{xx}(x, y) + u_{yy}(x, y) = f(x, y), & \text{for } (x, y) \in \Omega; \\ u(x, y) = g(x, y), & \text{for } (x, y) \in \partial\Omega, \end{cases} \tag{4}$$

   can have at most one solution $u \in C^2(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$.

   The PDE in (4),

   $$\Delta u = f, \qquad \text{in } \Omega,$$

   is called Poisson's equation. The BVP in (4) is then the Dirichlet problem for Poisson's equation in Ω.