Assignment #2

Due on Monday, February 11, 2019

Read Section 2.3, on *Extension of Solutions,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Do the following problems

1. Let U denote an open subset of $\mathbb{R}^N$, and $F\colon U \to \mathbb{R}^N$ be a C^1 vector field. For given $p \in U$, let $u_p\colon J_p \to U$ denote the unique solution to the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(x); \\ x(0) = p, \end{cases}$$

defined on a maximal interval of existence, J_p. For $s \in J_p$, put $q = u_p(s)$ and define

$$v(t) = u_p(t+s), \qquad \text{for all } t \in J_p - s,$$

where $J_p - s = \{t \in \mathbb{R} \mid t + s \in J_p\}$. Prove that v is the solution to the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(x); \\ \\ x(0) = q; \end{cases}$$

That is,

$$u_p(t+s) = u_q(t), \qquad \text{for all } t \in J_p - s,$$

where $q = u_p(s)$.

2. Find the maximal interval of existence, $J = (a, b)$, for the two–dimensional system

$$\begin{cases} \dfrac{dx}{dt} &= x^2; \\ \\ \dfrac{dy}{dt} &= y + \dfrac{1}{x}, \end{cases}$$

subject to the initial condition

$$\begin{cases} x(0) = 1; \\ y(0) = 1. \end{cases}$$

Compute the corresponding unique solution $u\colon J \to \mathbb{R}^2$. If either $a > -\infty$ or $b < \infty$, discuss the limit of $u(t)$ as $t \to a^+$, or $t \to b^-$, respectively.

3. Let I be an open interval and U an open subset of $\mathbb{R}^N$. Suppose that $F\colon I\times U \to \mathbb{R}^N$ is a continuous vector field which is bounded over $I \times U$. Let a and b be real numbers with $a < b$ and $[a,b] \subset I$, and suppose that
$$u\colon (a,b) \to U$$
is a solution to the equation $\dfrac{dx}{dt} = F(t,x)$. Prove that $\lim_{t\to a^+} u(t)$ and $\lim_{t\to b^-} u(t)$ exist.

Suggestion: Follow the following outline:

i. Let M be a positive number such that
$$\|F(t,x)\| \leqslant M, \qquad \text{for all } (t,x) \in I \times U,$$
and derive the estimate
$$\|u(t) - u(s)\| \leqslant M|t-s|, \quad \text{for all } t, s \in (a,b). \tag{1}$$

ii. Let (t_m) be any sequence in (a,b) which converges to b. Use the estimate in (1) to show that $(u(t_m)$ is a Cauchy sequence in $\mathbb{R}^N$. Therefore, the sequence of vectors, $(u(t_m))$, converges in $\mathbb{R}^N$ to some vector p.

iii. Let (t_m) and p be as in Part 3ii. Prove that
$$\lim_{t\to b^-} u(t) = p. \tag{2}$$
(Argue by contradiction. If (2) is not true, there is a positive number, ε, and sequence (s_m) in (a,b) such that $s_m \to b$ and
$$\|u(s_m) - p\| \geqslant \varepsilon.$$
Then, use (1) to estimate $\|u(t_m) - u(s_m)\|$.)

4. Let F, a, b and u be as in Problem 3, and suppose that $F(t,x)$ satisfies a local Lipschitz condition at every $(t,p) \in I \times U$. Prove that if $\lim_{t\to b^-} u(t) \in U$, then u can be extended to an interval $(a, b+\delta)$, for some $\delta > 0$.

State the analogous result at the endpoint a.

Suggestion: Let $p = \lim_{t\to b^-} u(t)$ and consider the IVP
$$\begin{cases} \dfrac{dx}{dt} = F(t,x); \\[2ex] x(b) = p. \end{cases}$$

5. The swings of a *simple (undamped) pendulum* are governed by the following second order ordinary differential equation (ODE):

$$m\ell\frac{d^2\theta}{dt^2} = -mg\sin\theta, \tag{3}$$

where $\theta = \theta(t)$ is a twice differentiable function which gives the angle the pendulum makes with a vertical line, m is the mass of the pendulum bob and ℓ is the length of the pendulum.

Introducing the new variables $x = \theta$ and $y = \dfrac{d\theta}{dt}$, the second order ODE in (3) can be turned into a two–dimensional system of first order equations of the form

$$\begin{cases} \dfrac{dx}{dt} = f(x,y); \\ \\ \dfrac{dy}{dt} = g(x,y), \end{cases} \tag{4}$$

for some real valued, C^1 functions, f and g.

(a) Give the functions f and g in the system in (4), and their respective domains in $\mathbb{R}^2$.

(b) Prove that solutions of (4) subject to the initial conditions

$$\begin{cases} x(0) = x_o; \\ y(0) = y_o, \end{cases}$$

exist for all $t \in \mathbb{R}$ and any $(x_o, y_o) \in \mathbb{R}^2$.

Suggestion: Apply an appropriate global existence result proved in Section 2.3 of the Class Lecture Notes.