Assignment #3

Due on Monday, February 25, 2019

Read Section 2.4 on *Continuous Dependence on Initial Conditions* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Do the following problems

1. Consider the one–dimensional system

$$\frac{dx}{dt} = F(x), \tag{1}$$

where $F\colon (0,\infty) \to \mathbb{R}$ is given by

$$F(x) = \frac{1}{2x}, \qquad \text{for all } x > 0. \tag{2}$$

(a) For $p > 0$, find the solution, $u_p\colon J_p \to \mathbb{R}$, of the equation in (1) subject to the initial condition

$$x(0) = p. \tag{3}$$

(b) Give the maximal interval of existence, J_p, for the solution, u_p, computed in part (a) of this problem. Write $J_p = (a, b)$. Compute

$$\lim_{t\to a^+} u_p(t)$$

and discuss your result in light of the Escape in Finite Time Theorem (Proposition 2.3.9) in the class lecture notes.

2. Let F be as in Problem 1. Denote by $\theta(t, p)$ the solution $u_p(t)$, for $t \in J_p$, of the IVP in (1) and (3).

(a) Give the domain of definition of θ and verify that θ is continuous on its domain.

(b) Verify that θ is also C^1 in its domain and compute the partial derivatives

$$\frac{\partial\theta}{\partial t}(t, p) \quad \text{and} \quad \frac{\partial\theta}{\partial p}(t, p).$$

3. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the vector field defined by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ 2y + x^2 \end{pmatrix}, \qquad \text{for all } x, y \in \mathbb{R}.$$

For every $p, q \in \mathbb{R}$, solve the IVP

$$\begin{cases} \begin{pmatrix} \dfrac{dx}{dt} \\ \\ \dfrac{dy}{dt} \end{pmatrix} = F\begin{pmatrix} x \\ y \end{pmatrix}; \\ \\ \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}, \end{cases} \tag{4}$$

Denote the solution of IVP (4) by $u_{(p,q)}(t)$, for t in a maximal interval of existence $J_{(p,q)}$.

(a) Give the maximal interval of existence $J_{(p,q)}$ for each $(p, q) \in \mathbb{R}^2$.

(b) Put $\theta(t, p, q) = u_{(p,q)}(t)$ for each $(p, q) \in \mathbb{R}^2$ and each $t \in J_{(p,q)}$. Give the domain of definition of the map θ and verify that θ is continuous in that domain.

4. Let F be as given in Problem 3 and let $\theta = \theta(t, p, q)$ denote the flow map for the field F, which was computed in part (b) of that problem. Verify that θ is a C^1 map and compute the derivative map,

$$D_{(p,q)}\theta(t, p, q)\colon U \to \mathbb{R}^2,$$

with respect to the initial points (p, q), for each $(p, q) \in \mathbb{R}^2$ and $t \in J_{(p,q)}$; more specifically, write

$$\theta(t, p, q) = \begin{pmatrix} f(t, p, q) \\ g(t, p, q) \end{pmatrix}$$

where f and g are real valued functions, and compute

$$D_{(p,q)}\theta(t, p, q) = \begin{pmatrix} \dfrac{\partial f}{\partial p}(t, p, q) & \dfrac{\partial f}{\partial q}(t, p, q) \\ \\ \dfrac{\partial g}{\partial p}(t, p, q) & \dfrac{\partial g}{\partial q}(t, p, q) \end{pmatrix}.$$

5. Let I denote an open interval and U an open subset of $\mathbb{R}^N$. Suppose that $F\colon I \times U \to \mathbb{R}^N$ is continuous Assume also that $F(t,x)$ satisfies a Lipschitz condition in x over a set $J \times V$, where J is an open subinterval of I, and V is an open subset of U; more specifically, assume that there exists a constant K such that

$$\|F(t,x) - F(t,y)\| \leqslant K\|x-y\|, \quad \text{for } x,y \in V, \text{ and } \ t \in J. \tag{5}$$

   (a) Explain why, for any $(t_o, p_o) \in J \times V$, the IVP

$$\begin{cases} \dfrac{dx}{dt} = F(t,x); \\ \\ x(t_o) = p_o. \end{cases} \tag{6}$$

   has a unique solution, $u_{p_o}\colon J_{p_o} \to U$, defined on some maximal interval J_{p_o} containing t_o.

   (b) Let $u_p\colon J_p \to U$ and $u_q\colon J_p \to U$ denote solutions of the differential equation

$$\frac{dx}{dt} = F(t,x)$$

   satisfying the

$$u_p(t_o) = p \quad \text{and} \quad u_q(t_o) = q$$

   where $t_o \in J$ and $p, q \in V$.
   Assume that

$$u_p(t) \in V \quad \text{and} \quad u_q(t) \in V, \quad \text{for all } t \in J.$$

   Prove that

$$\|u_p(t) - u_q(t)\| \leqslant \|p-q\| e^{K|t-t_o|}, \quad \text{for all } t \in J.$$