Assignment #11

Due on Friday, March 15, 2019

Read Section 4.3, on *Conservation of Momentum,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Do the following problems

1. Let $x\colon J \to \mathbb{R}$ demote a function that is twice–differentiable. Suppose that x solves the second order differential equations

$$\ddot{x} + a\dot{x} + bx = 0,$$

where a and b are real numbers.

By setting $y(t) = \dot{x}(t)$ for all $t \in J$, verify that the path $\sigma\colon J \to \mathbb{R}^2$ given by

$$\sigma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \qquad \text{for } t \in J,$$

solves the system of first–order differential equations

$$\begin{cases} \dot{x} &= y; \\ \dot{y} &= -bx - ay. \end{cases}$$

2. Let a and ω denote a positive numbers, and ϕ denote any real number. Define the path $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ by

$$\sigma(t) = a\begin{pmatrix} \sin(\omega t + \phi) \\ \omega\cos(\omega t + \phi) \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}. \tag{1}$$

Verify that $\sigma(t)$ solves the system of differentiable equations

$$\begin{cases} \dot{x} &= y; \\ \dot{y} &= -\omega^2 x. \end{cases} \tag{2}$$

3. Use the result of Problem 2 to sketch the phase portrait of the system in (2).

Consider the three cases: (i) $0 < \omega < 1$, (ii) $\omega = 1$, and (iii) $\omega > 1$.

4. Consider the second order differential equation

$$\ddot{x} = -\omega^2 x, \tag{3}$$

where ω is a positive number.

(a) Assume that $x\colon \mathbb{R} \to \mathbb{R}$ is a twice–differentiable function that solves the differential equation in (3), and set $y(t) = \dot{x}(t)$ for all $t \in \mathbb{R}$.

Verify that the path $\sigma\colon J \to \mathbb{R}^2$ given by

$$\sigma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \text{for } t \in R,$$

solves the system of differential equations in (2).

(b) Use the result of Problem 2 to obtain a solution of the second order differential equation (3) subject to the initial conditions $x(0) = x_o$ and $\dot{x}(0) = 0$, where x_o is a positive real number.

Sketch the solution.

5. Consider the second order differential equation

$$\ddot{x} = a^2 x, \tag{4}$$

where a is a positive number.

Define

$$x(t) = e^{\lambda t}, \quad \text{for } t \in \mathbb{R}. \tag{5}$$

(a) Determine distinct values of λ for which the function x defined in (5) solves the differential equation in (4).

(b) Let λ_1 and λ_2 denote the two distinct values of λ obtained in part (a).

Verify that the function $u\colon \mathbb{R} \to \mathbb{R}^2$ given by

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad \text{for } t \in \mathbb{R},$$

where c_1 and c_2 are constant, solves the differential equation in (4).