Solutions to Assignment #11

1. Let $x\colon J \to \mathbb{R}$ demote a function that is twice–differentiable. Suppose that x solves the second order differential equations

$$\ddot{x} + a\dot{x} + bx = 0, \tag{1}$$

where a and b are real numbers.

By setting $y(t) = \dot{x}(t)$ for all $t \in J$, verify that the path $\sigma\colon J \to \mathbb{R}^2$ given by

$$\sigma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \qquad \text{for } t \in J, \tag{2}$$

solves the system of first–order differential equations

$$\left\{ \begin{array}{rcl} \dot{x} & = & y; \\ \dot{y} & = & -bx - ay. \end{array} \right. \tag{3}$$

***Solution**:* Compute

$$\dot{y} = \frac{d}{dt}[\dot{x}] = \ddot{x};$$

so that, using (1),

$$\dot{y} = -a\dot{x} - bx,$$

which is the second equation in (3).

Since, $\dot{x} = y$, by the definition of y, the first equation in (3 is also satisfied. It then follows that the path in (2) solves the system in (3). $\square$

2. Let a and ω denote a positive numbers, and ϕ denote any real number. Define the path $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ by

$$\sigma(t) = a \begin{pmatrix} \sin(\omega t + \phi) \\ \omega\cos(\omega t + \phi) \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}. \tag{4}$$

Verify that $\sigma(t)$ solves the system of differentiable equations

$$\left\{ \begin{array}{rcl} \dot{x} & = & y; \\ \dot{y} & = & -\omega^2 x. \end{array} \right. \tag{5}$$

Solution: In this case, according to the definition of the path σ in (4),

$$\begin{cases} x(t) &= a\sin(\omega t + \phi); \\ & \\ y(t) &= a\omega\cos(\omega t + \phi), \end{cases} \qquad \text{for } t \in \mathbb{R}. \tag{6}$$

Taking the derivative with respect to t on both sides of the first equation in (6), we have

$$\dot{x}(t) = a\omega\cos(\omega t + \phi), \qquad \text{for } t \in \mathbb{R};$$

so that, according to the second equation in (6),

$$\dot{x} = y,$$

which is the first equation in (5).

Next, take the derivative with respect to t on both sides of the second equation in (6), to get

$$\dot{y}(t) = -a\omega^2\sin(\omega t + \phi), \qquad \text{for } t \in \mathbb{R};$$

so that, in view of the first equation of (6),

$$\dot{y} = -\omega^2 x,$$

which is the second equation in (5).

We have therefore shown that the path σ defined in (4) solves the system of differential equations in (5). $\square$

3. Use the result of Problem 2 to sketch the phase portrait of the system in (5).

   Consider the three cases: (i) $0 < \omega < 1$, (ii) $\omega = 1$, and (iii) $\omega > 1$.

   Solution: If $a = 0$ in the parametric equations in (6), we obtain the equilibrium solution $(0, 0)$, This solution is sketched in Figure 1, Figure 2 and Figure 3.

   Suppose $a > 0$ in the parametric equations in (6) and divide both equations by a to get

$$\begin{cases} \dfrac{x(t)}{a} &= \sin(\omega t + \phi); \\ & \\ \dfrac{y(t)}{\omega a} &= \cos(\omega t + \phi), \end{cases} \qquad \text{for } t \in \mathbb{R}. \tag{7}$$

   Thus, squaring on both sides of the equation in (7) and adding them, we get

$$\frac{x^2}{a^2} + \frac{y^2}{\omega^2 a^2} = 1, \tag{8}$$

where we have used the trigonometric identity

$$\cos^2 A + \sin^2 A = 1.$$

The graph of the equation in (8) is a circle of radius a centered at the origin in the case $\omega = 1$. In the case $\omega \neq 1$, the graph is an ellipse with vertices $(-a, 0)$ and $(a, 0)$ on the x–axis and vertices $(0, -\omega a)$ and $(0, \omega a)$ on the y–axis. We sketch the phase portrait of the system in (5) for each of the cases (i) $0 < \omega < 1$, (ii) $\omega = 1$, and (iii) $\omega > 1$, separately

(i) Figure 1 shows a sketch of the phase portrait of the system in (5) for the case $0 < \omega < 1$. The sketch also shows the direction along the orbits

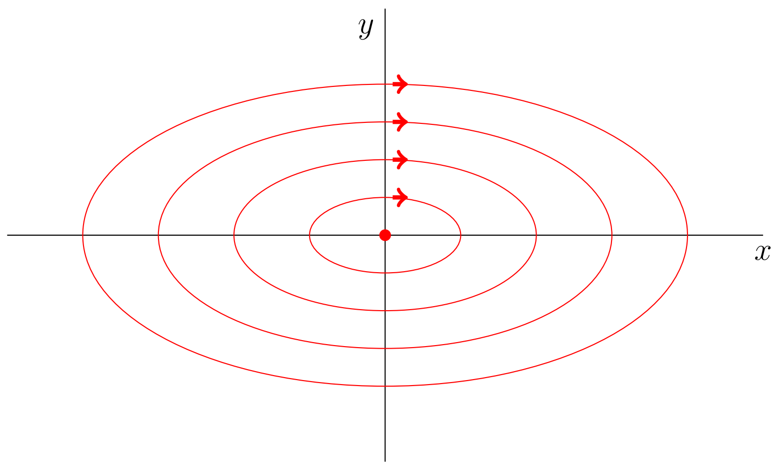

Figure 1: Sketch of phase portrait of the system in (5) for $0 < \omega < 1$

dictated by the system of differential equations in (5). For instance, in the first quadrant, since $x > 0$ and $y > 0$, we get from the equations in (5) that $\dot{x} > 0$ and $\dot{y} < 0$; thus, the direction along the ellipses is in the clockwise sense.

(ii) In the case $\omega = 1$, the phase portrait of the system in (5) consists of concentric circles centered at the origin oriented in the clockwise sense. A sketch of this situation is shown in Figure 2.

(iii) The sketch in Figure 3 shows a few of those ellipses for varies values of $a > 0$ in the case $\omega > 1$.

$\square$

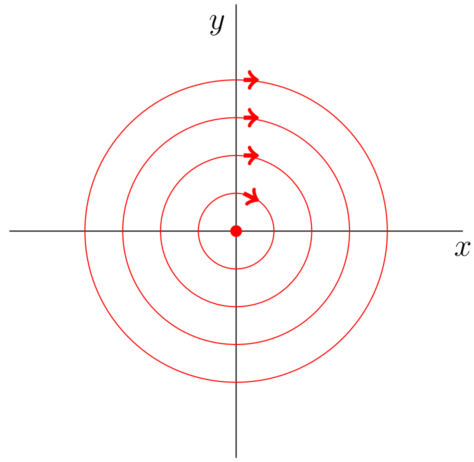

Figure 2: Sketch of phase portrait of the system in (5) for $0\omega = 1$

4. Consider the second order differential equation

$$\ddot{x} = -\omega^2 x, \tag{9}$$

where ω is a positive number.

(a) Assume that $x\colon \mathbb{R} \to \mathbb{R}$ is a twice–differentiable function that solves the differential equation in (9), and set $y(t) = \dot{x}(t)$ for all $t \in \mathbb{R}$.
Verify that the path $\sigma\colon J \to \mathbb{R}^2$ given by

$$\sigma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \text{for } t \in R, \tag{10}$$

solves the system of differential equations in (5).
***Solution**:* Suppose that $x\colon \mathbb{R} \to \mathbb{R}$ is a twice–differentiable function that solves the differential equation in (9), and set $y(t) = \dot{x}(t)$ for all $t \in \mathbb{R}$.
Compute

$$\dot{y} = \frac{d}{dt}[\dot{x}] = \ddot{x};$$

so that, in view of (9),

$$\dot{y} = -\omega^2 x,$$

which is the second equation in the system in (5).
Since, $\dot{x} = y$, by the definition of y, the first equation in (5) is also satisfied.
Hence, the path σ defined in (10) solves the system of differential equations in (5). □

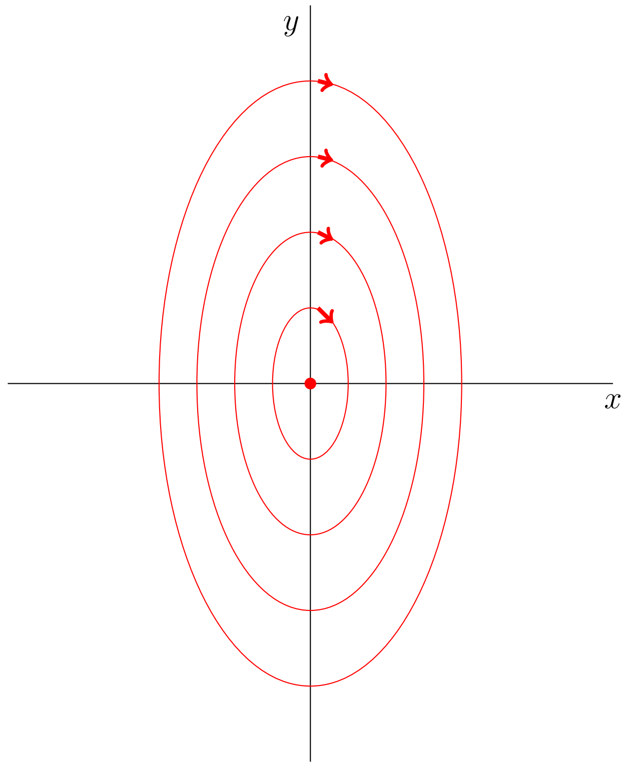

Figure 3: Sketch of phase portrait of the system in (5) for $\omega > 1$

(b) Use the result of Problem 2 to obtain a solution of the second order differential equation (9) subject to the initial conditions $x(0) = x_o$ and $\dot{x}(0) = 0$, where x_o is a positive real number.

Sketch the solution.

Solution*:* This problem can be stated as the following initial value problem (IVP):

$$\begin{cases} \ddot{x} = -\omega^2 x; \\ x(0) = x_o; \\ \dot{x}(0) = 0. \end{cases} \tag{11}$$

Let x denote a solution of the differential equation in (11). By the result in part (a) of this problem, setting $y = \dot{x}$, the path σ defined in (10) solves the system in (5).

It was shown in Problem 2 that the path

$$\sigma(t) = \begin{pmatrix} a\sin(\omega t + \phi) \\ a\omega\cos(\omega t + \phi) \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}, \tag{12}$$

solves the system in (5). Thus, a solution of the second–order differential equation in (11) is given by the first component of the path in (12); namely,

$$x(t) = a\sin(\omega t + \phi), \qquad \text{for } t \in \mathbb{R}, \tag{13}$$

where a and ϕ are constants.

We next determine values of a and ϕ so that the initial conditions in the IVP in (11) are satisfied.

From (13) we obtain that

$$\dot{x}(t) = a\omega\cos(\omega t + \phi), \qquad \text{for } t \in \mathbb{R}, \tag{14}$$

where we have used the Chain Rule.

Substitute 0 for t in (13) and (14), and use the initial conditions in (11) to get

$$\left\{ \begin{array}{rcl} a\sin(\phi) & = & x_o; \\ a\omega\cos(\phi) & = & 0. \end{array} \right. \tag{15}$$

We first note that a cannot be 0; otherwise, $x(t) = 0$, for all t, according to (13), and this is incompatible with the initial condition $x_o > 0$. Hence, since we are also assuming that $\omega > 0$, we get from the second equation in (15) that

$$\cos(\phi) = 0;$$

thus, we can take

$$\phi = \frac{\pi}{2}. \tag{16}$$

Substituting the value of ϕ in (16) into the first equation in (15) then yields

$$a = x_o. \tag{17}$$

Substitute the values for a and ϕ in (17) and (16), respectively, into the formula for $x(t)$ in (13) to get

$$x(t) = x_o \sin\left(\omega t + \frac{\pi}{2}\right), \qquad \text{for } t \in \mathbb{R},$$

which, using the trigonometric identity

$$\sin(A + B) = \sin A \cos B + \cos A \sin B,$$

can be rewritten as

$$x(t) = x_o\cos(\omega t), \qquad \text{for } t \in \mathbb{R}. \tag{18}$$

A sketch of the function in (18) is shown in Figure 4. $\square$

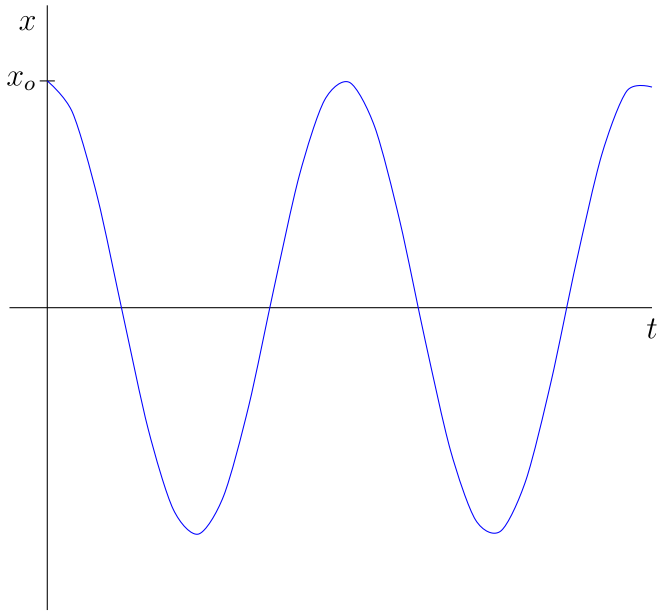

Figure 4: Sketch of x as a function of t

5. Consider the second order differential equation

$$\ddot{x} = a^2 x, \tag{19}$$

where a is a positive number.

Define

$$x(t) = e^{\lambda t}, \quad \text{for } t \in \mathbb{R}. \tag{20}$$

(a) Determine distinct values of λ for which the function x defined in (20) solves the differential equation in (19).

***Solution**:* Differentiate the function in (20) with respect to t to get

$$\dot{x}(t) = \lambda e^{\lambda t}, \quad \text{for } t \in \mathbb{R}. \tag{21}$$

Similarly, differentiating the function in (21 with respect to t yields

$$\ddot{x}(t) = \lambda^2 e^{\lambda t}, \quad \text{for } t \in \mathbb{R}. \tag{22}$$

Next, substitute the functions in (22) and (20) into the second–order differential equation in (19) to get

$$\lambda^2 e^{\lambda t} = a^2 e^{\lambda t}, \quad \text{for } t \in \mathbb{R};$$

so that, since the exponential function is never 0,

$$\lambda^2 = a^2. \tag{23}$$

The equation in (23) has solutions

$$\lambda_1 = -a \quad \text{and} \quad \lambda_2 = -a. \tag{24}$$

□

(b) Let λ_1 and λ_2 denote the two distinct values of λ obtained in part (a). Verify that the function $u\colon \mathbb{R} \to \mathbb{R}^2$ given by

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad \text{for } t \in \mathbb{R},$$

where c_1 and c_2 are constant, solves the differential equation in (19).
Solution: With the values of λ in (24), we have that

$$u(t) = c_1 e^{-at} + c_2 e^{at}, \quad \text{for } t \in \mathbb{R} \tag{25}$$

Differentiate the function u in (25) to get

$$\dot{u}(t) = -ac_1 e^{-at} + ac_2 e^{at}, \quad \text{for } t \in \mathbb{R}, \tag{26}$$

where we have used the Chain Rule.
Similarly, differentiating with respect to t the function in (26),

$$\ddot{u}(t) = a^2 c_1 e^{-at} + a^2 c_2 e^{at}, \quad \text{for } t \in \mathbb{R}. \tag{27}$$

Factoring a^2 in the right–hand side of (27) we get

$$\ddot{u}(t) = a^2 (c_1 e^{-at} + c_2 e^{at}), \quad \text{for } t \in \mathbb{R};$$

so that, in view of the definition of u in (25),

$$\ddot{u} = a^2 u,$$

which shows that u solves the differential equation in (19). □