Solutions to Assignment #12

1. Let A be the 2×2 matrix given by $A = \begin{pmatrix} -1 & 1 \\ 5 & -1 \end{pmatrix}$. Let v and w denote the column vectors $\mathrm{v} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ and $\mathrm{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Compute $A\mathrm{v}$ and $A\mathrm{w}$.

   ***Solution**:* Compute

$$A\mathrm{v} = \begin{pmatrix} -1 & 1 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix};$$

   and

$$A\mathrm{w} = \begin{pmatrix} -1 & 1 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

□

2. Let A, v and w be as in Problem 1. Compute the vector $2\mathrm{v} - 3\mathrm{w}$ and compute the product $A(2\mathrm{v} - 3\mathrm{w})$. Verify that

$$A(2\mathrm{v} - 3\mathrm{w}) = 2A\mathrm{v} - 3A\mathrm{w}.$$

   ***Solution**:* Compute

$$\begin{aligned} 2\mathrm{v} - 3\mathrm{w} &= 2\begin{pmatrix} 1 \\ 5 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 10 \end{pmatrix} + \begin{pmatrix} -3 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 7 \end{pmatrix}; \end{aligned}$$

$$A(2\mathrm{v} - 3\mathrm{w}) = \begin{pmatrix} -1 & 1 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 7 \end{pmatrix} = \begin{pmatrix} 8 \\ -12 \end{pmatrix}; \qquad (1)$$

   and

$$\begin{aligned} 2A\mathrm{v} - 3A\mathrm{w} &= 2\begin{pmatrix} 4 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 8 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -12 \end{pmatrix}; \end{aligned}$$

so that,

$$2A\text{v} - 3A\text{w} = \begin{pmatrix} 8 \\ -12 \end{pmatrix}. \tag{2}$$

Comparing (1) and (2) we see that

$$A(2\text{v} - 3\text{w}) = 2A\text{v} - 3A\text{w}.$$

□

3. Find a condition on the scalars a, b, c and d so that the columns of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

are not scalar multiples of each other; that is the column vectors of A do not lie on the same line.

Suggestion: Consider the cases $a = 0$ and $a \neq 0$ separately.

***Solution**:* Let

$$\text{v}_1 = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad \text{v}_2 = \begin{pmatrix} b \\ d \end{pmatrix}.$$

Firs, consider the case in which $a \neq 0$ and $b \neq 0$. In this case, the vectors v_1 and v_2 do not lie on the same line is the slopes

$$m_1 = \frac{c}{a} \quad \text{and} \quad m_2 = \frac{d}{b}$$

are not the same; that is,

$$m_2 \neq m_1,$$

or

$$\frac{d}{b} \neq \frac{c}{a},$$

or

$$ad \neq bd,$$

or

$$ad - bc \neq 0. \tag{3}$$

Thus, we see that if (3) and $a \neq 0$ and $b \neq 0$, the vectors v_1 and v_2 cannot be parallel.

On the other hand, if (3) holds true and $a = 0$; then,

$$bc \neq 0. \tag{4}$$

Thus,

$$\mathrm{v}_1 = \begin{pmatrix} 0 \\ c \end{pmatrix} \quad \text{and} \quad \mathrm{v}_2 = \begin{pmatrix} b \\ d \end{pmatrix},$$

where $c \neq 0$ and $b \neq 0$ by virtue of (4). Hence, v_1 lies along the y–axis and does not, since $b \neq 0$.

By the same token, if (3) holds true and $b = 0$, then

$$ad \neq 0; \tag{5}$$

so that,

$$a \neq 0 \quad \text{and} \quad d \neq 0.$$

Thus,

$$\mathrm{v}_1 = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad \mathrm{v}_2 = \begin{pmatrix} 0 \\ d \end{pmatrix},$$

which implies that v_2 is parallel to the y–axis, but v_1 is not. □

4. Let the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfy the condition you discovered in Problem 3. Show that the matrix equation $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has only one solution; namely, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

***Solution**:* Assume that the condition (3) holds true; that is, assume that

$$ad - bc \neq 0, \tag{6}$$

and consider the equation

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{7}$$

which is equivalent to the system of equations

$$\left\{ \begin{array}{rcl} ax + by & = & 0; \\ cx + dy & = & 0. \end{array} \right. \tag{8}$$

We consider two cases: (i) $a \neq 0$, and (ii) $a = 0$.

(i) If $a \neq 0$, then we can solve the first equation in (8) for x to get

$$x = -\frac{b}{a}y. \tag{9}$$

Then, substitute the result in (9) into the second equation in (8) to get

$$c\left(-\frac{b}{a}y\right) + dy = 0,$$

which can be rewritten as

$$\frac{ad - bc}{a}y = 0. \tag{10}$$

Since we are assuming that (6) holds true, it follows from (10) that

$$y = 0. \tag{11}$$

Next, substitute the result in (11) into (9) to get

$$x = 0. \tag{12}$$

In view of (11) and (12) we see that, if $ad - bc \neq 0$, then $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the only solution of the equation in (7).

(ii) Suppose that $a = 0$ and $ad - bc \neq 0$; so that

$$bc \neq 0,$$

from which we get that

$$b \neq 0 \quad \text{and} \quad c \neq 0. \tag{13}$$

It then follows from the first equation in (8) that

$$by = 0;$$

so that, by virtue of the first condition in (13),

$$y = 0. \tag{14}$$

Next, substitute the result in (14) into the second equation in (8) that

$$cx = 0,$$

so that, in view of the second condition in (13),

$$x = 0. \tag{15}$$

Combining (14) and (15) we see that, also in this case, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the only solution of the equation in (7).

□

5. Let A denote the matrix in Problem 1. Let v_1 denote the first column of A and v_2 denote the second column of A. Find scalars c_1 and c_2 for which

$$c_1\mathrm{v}_1 + c_2\mathrm{v}_2 = \begin{pmatrix} 4 \\ 7 \end{pmatrix}. \tag{16}$$

***Solution**:* Rewrite the equation in (16) as

$$c_1 \begin{pmatrix} -1 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix},$$

or

$$\begin{pmatrix} -c_1 \\ 5c_1 \end{pmatrix} + \begin{pmatrix} c_2 \\ -c_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix},$$

or

$$\begin{pmatrix} -c_1 + c_2 \\ 5c_1 - c_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix};$$

so that, the equation in (16) is equivalent to the system of equations

$$\begin{cases} -c_1 + c_2 &= 4; \\ 5c_1 - c_2 &= 7. \end{cases} \tag{17}$$

Adding the equations in (17) yields

$$4c_1 = 11,$$

from which we get that

$$c_1 = \frac{11}{4}. \tag{18}$$

Substitute the result in (18) into the first equation in (17) yields

$$-\frac{11}{4} + c_2 = 4,$$

which can be solved for c_2 to get

$$c_2 = \frac{27}{4}.$$

□