Solutions to Assignment #13

1. Let A be the 2×2 matrix given by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $ad - bc \neq 0$.

   Set $\Delta = ad - bc$ and define $B = \dfrac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Verify that $AB = BA = I$, where I denotes the 2×2 identity matrix.

   Solution: Compute

$$
\begin{aligned}
AB &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\
&= \frac{1}{\Delta}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\
&= \frac{1}{\Delta}\begin{pmatrix} ab - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\
&= \frac{1}{\Delta}\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix},
\end{aligned}
$$

   from which we get that $AB = I$.

   Similarly,

$$
\begin{aligned}
BA &= \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \frac{1}{\Delta}\begin{pmatrix} ab - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\
&= \frac{1}{\Delta}\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix},
\end{aligned}
$$

   from which we get $BA = I$. □

2. Let $A = \begin{pmatrix} -1 & 4 \\ -2 & 3 \end{pmatrix}$. Use the result in Problem 1 to find a matrix B such that $AB = BA = I$, where I denotes the 2×2 identity matrix.

Solution: In this case, $\Delta = (-1)(3) - (-2)(4) = 5$. So, $\Delta \neq 0$. Thus, according to the result on Problem 1, A has an inverse given by

$$A^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix},$$

or

$$A^{-1} = \begin{pmatrix} 3/5 & -4/5 \\ 2/5 & -1/5 \end{pmatrix}.$$

□

3. Let A be the matrix given in Problem 2. Compute $A^2 - 2A + 5I$, where I denotes the 2×2 identity matrix.

Solution: Compute

$$\begin{aligned} A^2 - 2A + 5I &= \begin{pmatrix} -1 & 4 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ -2 & 3 \end{pmatrix} - 2\begin{pmatrix} -1 & 4 \\ -2 & 3 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -7 & 8 \\ -4 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -8 \\ 4 & -6 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

the 2×2 zero matrix, O. Thus, $A^2 - 2A + 5I = O$. □

4. Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$, let $\mathrm{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Compute the product $A\mathrm{v}_1$. What do you conclude?

Solution: Compute

$$A\mathrm{v}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix};$$

so that, $A\mathrm{v}_1 = \mathrm{v}_1$. Hence, A maps v_1 to itself.

Alternatively, v_1 is an eigenvector of the matrix A associated with the eigenvalue $\lambda_1 = 1$. □

5. Let A and v_1 be as given in Problem 4. Find all vectors $\mathrm{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ such that

$$(A - I)\mathrm{v} = \mathrm{v}_1,$$

where I denotes the 2×2 identity matrix.

Solution: Solve the matrix equation

$$\left(\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

or

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

or

$$\begin{pmatrix} -x - y \\ x + y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{1}$$

The vector equation in (1) is equivalent to the equation

$$x + y = -1. \tag{2}$$

Solving for x in the equation (2) we obtain

$$x = -y - 1. \tag{3}$$

Setting $y = -t$, where t is a real parameter, we obtain that parametric equations

$$\begin{cases} x = t - 1; \\ y = -t. \end{cases} \tag{4}$$

Thus, the vectors $\mathrm{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ that solve the equation

$$(A - I)\mathrm{v} = \mathrm{v}_1,$$

are of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t - 1 \\ -t \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

according to (4). □