Assignment #14

Due on Wednesday, April 10, 2019

Read Section 5.3, on *The Flow of TwoDimensional Linear Fields,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Do the following problems

1. Let A be the 2×2 matrix and suppose that v is a nonzero vector in $\mathbb{R}^2$ such that
$$A\text{v} = \lambda\text{v}, \tag{1}$$
for some scalar λ.

   Define the path $\begin{pmatrix} x \\ y \end{pmatrix} : \mathbb{R} \to \mathbb{R}^2$ by
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = ce^{\lambda t}\text{v}, \text{ for all } t \in \mathbb{R}, \tag{2}$$
where c is scalar constant. Verify that $\begin{pmatrix} x \\ y \end{pmatrix}$ is a solution of the system of first order differential equations
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \tag{3}$$
where the dot above the variable name indicates derivative with respect to t.

   Suggestion: Differentiate on both sides of (2) with respect to t and use (1).

   Notation. The function in (2) is called a line solution of the system in (3).

2. Let A denote the 2×2 matrix
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
and let $\text{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\text{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

   Verify that $A\text{v}_1 = \lambda_1\text{v}_1$, where $\lambda_1 = -1$; and $A\text{v}_2 = \lambda_2\text{v}_2$, where $\lambda_2 = 1$.

3. Consider the system

$$\begin{cases} \dfrac{dx}{dt} = y; \\ \\ \dfrac{dy}{dt} = x. \end{cases} \tag{4}$$

   (a) Show that the system in (4) can be written in vector form as in (3) where A is the matrix given in Problem 2.

   (b) Let v_1 and v_2 be the vectors given in Problem 2, $\lambda_1 = -1$ and $\lambda_2 = 1$. Use the result in Problem 1 to show that

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = e^{\lambda_1 t}\mathrm{v}_1 \quad \text{and} \quad \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = e^{\lambda_2 t}\mathrm{v}_2, \quad \text{for all } t \in \mathbb{R},$$

   define solutions of the system in (4).

4. Let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be the paths defined in Problem 3. Verify that the function $\begin{pmatrix} x \\ y \end{pmatrix} : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} + c_2 \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}, \quad \text{for all } t \in \mathbb{R}, \tag{5}$$

   solves the system in (4).

5. Use the function given in (5) to sketch the flow of the vector field

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$