Solutions to Assignment #15

1. Let A be the 2×2 matrix $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$. Find all eigenvalues of A and give corresponding eigenvectors.

***Solution**:* The characteristic polynomial of A is

$$p_{_A}(\lambda) = \lambda^2 - 3\lambda + 2,$$

which factors into

$$p_{_A}(\lambda) = (\lambda - 1)(\lambda - 2).$$

Thus, the eigenvalues of A are

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = 2.$$

To find an eigenvalue corresponding to λ_1, solve the system equations

$$(A - \lambda_1 I) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where I denotes the 2×2 identity matrix, or

$$\left\{ \begin{array}{rcl} (0-1)x - 2y & = & 0; \\ x + (3-1)y & = & 0, \end{array} \right.$$

or

$$\left\{ \begin{array}{rcl} -x - 2y & = & 0; \\ x + 2y & = & 0, \end{array} \right.$$

which reduces to the single equation

$$x + 2y = 0. \tag{1}$$

We find all solutions of the equation in (1) by solving for x,

$$x = -2y,$$

and setting $y = -t$, where t is a parameter.

We then have that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2t \\ -t \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}. \tag{2}$$

Taking $t = 1$ in (2) yields

$$\mathrm{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

This is an eigenvector corresponding to the eigenvalue $\lambda_1 = 1$.

Similarly, solving the equation

$$(A - \lambda_2 I) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

leads to the system of equations

$$\begin{cases} (0-2)x - 2y &= 0; \\ x + (3-2)y &= 0, \end{cases}$$

or

$$\begin{cases} -2)x - 2y &= 0; \\ x + y &= 0, \end{cases}$$

which reduces to the equation

$$x + y = 0. \tag{3}$$

Solve for x in (3),

$$x = -y,$$

and set $y = -t$, where t is a parameter, to get the solutions

$$\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}, \tag{4}$$

of the equation in (3.

Taking $t = 1$ in (4) yields

$$\mathrm{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This is an eigenvector corresponding to the eigenvalue $\lambda_2 = 2$. □

2. Let A be the 2×2 matrix $A = \begin{pmatrix} 0 & -4 \\ 1 & 4 \end{pmatrix}$. Find all eigenvalues of A and give corresponding eigenvectors.

***Solution**:* The characteristic polynomial of the matrix A is

$$p_{_A}(\lambda) = \lambda^2 - \text{trace}(A)\lambda + \det(A),$$

where $\text{trace}(A) = 4$ and $\det(A) = 4$. Thus,

$$p_{_A}(\lambda) = \lambda^2 - 4\lambda + 4,$$

which factors into

$$p_{_A}(\lambda) = (\lambda - 2)^2,$$

Consequently, the matrix A has only one eigenvalue

$$\lambda = 2.$$

To find a corresponding eigenvalue, solve the system equations

$$(A - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where I denotes the 2×2 identity matrix, or

$$\begin{cases} (0-2)x - 4y &= 0; \\ x + (4-2)y &= 0, \end{cases}$$

or

$$\begin{cases} -2x - 4y &= 0; \\ x + 2y &= 0, \end{cases}$$

which reduces to the single equation

$$x + 2y = 0. \tag{5}$$

We find all solutions of the equation in (5) by solving for x,

$$x = -2y,$$

and setting $y = -t$, where t is a parameter.

We then have that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2t \\ -t \end{pmatrix}, \quad \text{for } t \in \mathbb{R},$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \qquad \text{for } t \in \mathbb{R}. \tag{6}$$

Taking $t = 1$ in (6) yields

$$\mathrm{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

This is an eigenvector corresponding to the eigenvalue $\lambda = 2$. □

3. Suppose that a 2×2 matrix A has real eigenvalues, λ_1 and λ_2, with $\lambda_1 \neq \lambda_2$. Let v_1 be an eigenvector corresponding to the eigenvalue λ_1, and v_2 be an eigenvector corresponding to the eigenvalue λ_2. Show that v_1 and v_2 cannot be multiples of each other.

   ***Solution**:* Assume, to the contrary, that v_2 is a scalar multiple of v_1; so that,

$$\mathrm{v}_2 = c\mathrm{v}_1, \tag{7}$$

for some scalar c. Then,

$$c \neq 0; \tag{8}$$

otherwise, in view of (7), v_2 would be the zero vector. But this is impossible because v_2 is an eigenvector of A.

Multiply both sides of the equation in (7) on the left by the matrix A to get

$$A\mathrm{v}_2 = A(c\mathrm{v}_1),$$

$$A\mathrm{v}_2 = cA\mathrm{v}_1,$$

or

$$\lambda_2\mathrm{v}_2 = c\lambda_1\mathrm{v}_1 \tag{9}$$

because v_1 is an eigenvector of A corresponding to λ_1, and v_2 is an eigenvector of A corresponding to λ_2.

Since we are given that $\lambda_1 \neq \lambda_2$, we may assume that $\lambda_2 \neq 0$. Thus, we can multiply the equation in (7) by λ_2 to get

$$\lambda_2\mathrm{v}_2 = c\lambda_2\mathrm{v}_1. \tag{10}$$

Comparing the equations in (9) and (10) we see that

$$c\lambda_2\mathrm{v}_1 = c\lambda_1\mathrm{v}_1,$$

from which we get that

$$c(\lambda_2 - \lambda_1)\mathrm{v}_1 = \mathbf{0}, \tag{11}$$

where $\mathbf{0}$ denotes the zero–vector in $\mathbb{R}^2$.

Since $\mathrm{v}_1 \neq \mathbf{0}$ because v_1 is an eigenvector of A, It follows from (11) that

$$c(\lambda_2 - \lambda_1) = 0. \tag{12}$$

It follows from (12) and the assumption that $\lambda_1 \neq \lambda_2$ that

$$c = 0,$$

which is in direct contradiction with (8). Therefore, (7) is impossible if v_1 and v_2 are eigenvectors of A corresponding to distinct eigenvalues λ_1 and λ_2, respectively. □

4. In this problem and the next we come up with solutions to the system

$$\begin{cases} \dot{x} &= \alpha x - \beta y; \\ \dot{y} &= \beta x + \alpha y, \end{cases} \tag{13}$$

where $\alpha^2 + \beta^2 \neq 0$ and $\beta \neq 0$.

Make the change of variables $x = r\cos\theta$ and $y = r\sin\theta$.

(a) Verify that $r^2 = x^2 + y^2$ and $\tan\theta = \dfrac{y}{x}$, provided that $x^2 + y^2 \neq 0$ and $x \neq 0$.

***Solution**:* Compute

$$\begin{aligned} x^2 + y^2 &= (r\cos\theta)^2 + (r\sin\theta)^2 \\ &= r^2\cos^2\theta + r^2\sin^2\theta \\ &= r^2(\cos^2\theta + \sin^2\theta) \\ &= r^2, \end{aligned}$$

from which we get that

$$r^2 = x^2 + y^2. \tag{14}$$

Similarly, assuming that $r \neq 0$ and $x = r\cos\theta \neq 0$,

$$\begin{aligned} \frac{y}{x} &= \frac{r\sin\theta}{r\cos\theta} \\ &= \frac{\sin\theta}{\cos\theta} \\ &= \tan\theta, \end{aligned}$$

from which we get that

$$\tan\theta = \frac{y}{x}, \qquad \text{for } x \neq 0 \text{ and } r \neq 0. \tag{15}$$

□

(b) Verify that

$$\begin{cases} \dot{r} = \dfrac{x\dot{x} + y\dot{y}}{r}, \\ \dot{\theta} = \dfrac{x\dot{y} - y\dot{x}}{r^2}. \end{cases} \tag{16}$$

***Solution**:* Take the derivative with respect to t on both sides of the equation in (14), and apply the Chain Rule, to get that

$$2r\dot{r} = 2x\dot{x} + 2y\dot{y},$$

from which we get that

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r},$$

which is the first equation in (16).

Next, take the derivative with respect to t on both sides of the equation in (15), using the Chain Rule, to get

$$(\sec^2\theta)\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{x^2}, \tag{17}$$

where

$$\sec^2\theta = 1 + \tan^2\theta,$$

or, in view of (15),

$$\begin{aligned} \sec^2\theta &= 1 + \frac{y^2}{x^2} \\ &= \frac{x^2 + y^2}{x^2}; \end{aligned}$$

so that,

$$\sec^2\theta = \frac{r^2}{x^2}. \tag{18}$$

where we have used (14).

Next, substitute the expression for $\sec^2\theta$ in (18) into the left–hand side of (17) to get

$$\frac{r^2}{x^2}\dot\theta = \frac{x\dot y - y\dot x}{x^2},$$

from which we get

$$\dot\theta = \frac{x\dot y - y\dot x}{r^2},$$

which is the second equation in (16). □

5. *[Problem 4 Continued]*

   (a) Use the result in (16) to transform the system (13) into a system involving r and θ.

   ***Solution**:* Substituting the expressions for $\dot x$ and $\dot y$ in the system in (13) into the first equation in (16) yields

$$\begin{aligned}\dot r &= \frac{x(\alpha x - \beta y) + y(\beta x + \alpha y)}{r}\\ &= \frac{\alpha x^2 - \beta xy + \beta xy + \alpha y^2}{r}\\ &= \frac{\alpha(x^2+y^2)}{r};\end{aligned}$$

   so that, in view of (14),

$$\dot r = \alpha r. \tag{19}$$

   Next, substitute the expressions for $\dot x$ and $\dot y$ in the system in (13) into the second equation in (16) to get

$$\begin{aligned}\dot\theta &= \frac{x(\beta x + \alpha y - y(\alpha x - \beta y)}{r^2}\\ &= \frac{\beta x^2 + \alpha xy - \alpha xy + \beta y^2}{r^2}\\ &= \frac{\beta(x^2+y^2)}{r^2};\end{aligned}$$

so that, by virtue of (14),

$$\dot{\theta} = \beta. \tag{20}$$

Putting together the equations in (19) and (20) yields the system

$$\begin{cases} \dot{r} & = & \alpha r; \\ \dot{\theta} & = & \beta. \end{cases} \tag{21}$$

□

(b) Solve the system obtained in part (a) of Problem 5 for r and θ.

***Solution**:* The first differential equation in (21) can be solved by separation of variables to yield

$$r(t) = Ce^{\alpha t}, \quad \text{for } t \in \mathbb{R}, \tag{22}$$

where C is a constant of integration.

The second differential equation in in (21) can be integrated to yield

$$\theta(t) = \beta t + \phi, \quad \text{for } t \in \mathbb{R}, \tag{23}$$

where ϕ is a constant of integration. □

(c) Based on your solution in part (b), give the general solution or the system (13).

***Solution**:* Using the change of variables $x = r\cos\theta$ and $y = r\sin\theta$, we obtain from (22) and (23) that

$$\begin{cases} x(t) & = & Ce^{\alpha t}\cos(\beta t + \phi); \\ y(t) & = & Ce^{\alpha t}\sin(\beta t + \phi), \end{cases} \quad \text{for } t \in \mathbb{R},$$

which we can write in vector form as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = Ce^{\alpha t}\begin{pmatrix} \cos(\beta t + \phi) \\ \sin(\beta t + \phi) \end{pmatrix}, \quad \text{for } t \in \mathbb{R}, \tag{24}$$

□

(d) Sketch the flow of the vector field associated with the system in (13) for $\beta = 1$ and each of the following cases

(i) $\alpha < 0$;

(ii) $\alpha = 0$; and

(iii) $\alpha > 0$.

Solution: Setting $\beta = 1$ in (24), we obtain

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = Ce^{\alpha t} \begin{pmatrix} \cos(t+\phi) \\ \sin(t+\phi) \end{pmatrix}, \quad \text{for } t \in \mathbb{R}, \tag{25}$$

as the general solution of the system of differential equations in (13).

We sketch a few of the curves parametrized by the vector valued function in (25) for each of the possibilities (i) $\alpha < 0$, (ii) $\alpha = 0$ and (iii) $\alpha > 0$, and various values for C and ϕ.

In all cases,

$$\left\| \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \right\| = r(t) = Ce^{\alpha t}, \quad \text{for } t \in \mathbb{R}, \tag{26}$$

according to (22).

(i) It follows from (26) that, if $\alpha < 0$, the distance from the point $(x(t), y(t))$ in the solution curve to the origin decreases as time increases. At the same time, the angle that the vector from $(0, 0)$ to $(x(t), y(t))$ makes with the positive axis also increases according to (23), since we are assuming that $\beta = 1$. Hence, in addition to the equilibrium solution at the origin, the solution curves spiral towards the origin in the counterclockwise direction as depicted in Figure 1.

(ii) If $\alpha = 0$ in (13), and $\beta = 1$, then, according to (22), or (26),

$$\left\| \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \right\| = C, \quad \text{for } t \in \mathbb{R}; \tag{27}$$

so that, the orbits of the system in (13) in this case include the origin and concentric circles around the origin traversed in the counterclockwise sense. A few of these trajectories are shown in Figure 2.

(iii) In the case $\alpha > 0$ and $\beta = 1$, the trajectories of the system in (13) include the origin and curves that spiral away from the origin in the counterclockwise sense. A few of these trajectories are sketched in Figure 3.

□

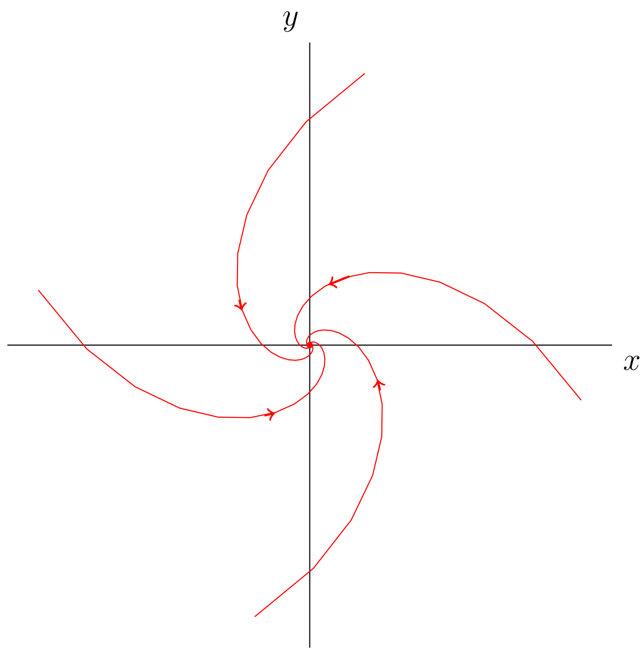

Figure 1: Sketch of phase portrait of system (13) with $\alpha < 0$ and $\beta = 1$

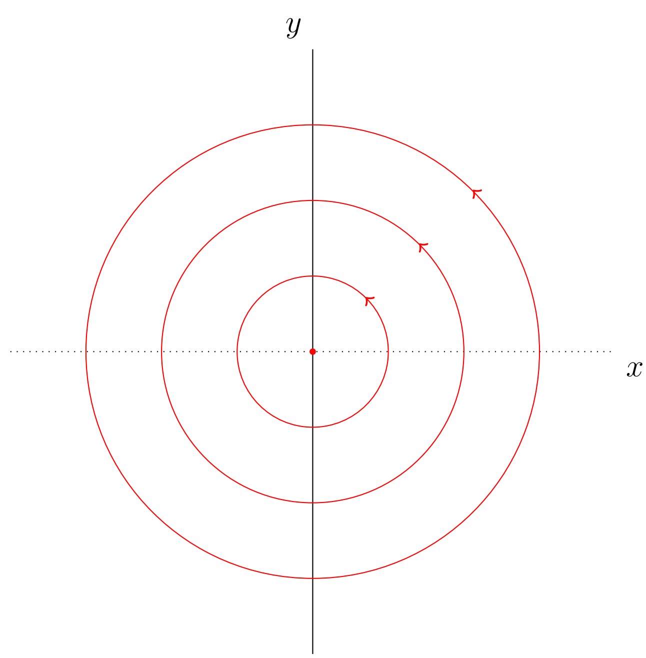

Figure 2: Sketch of phase portrait of system (13) with $\alpha = 0$ and $\beta = 1$

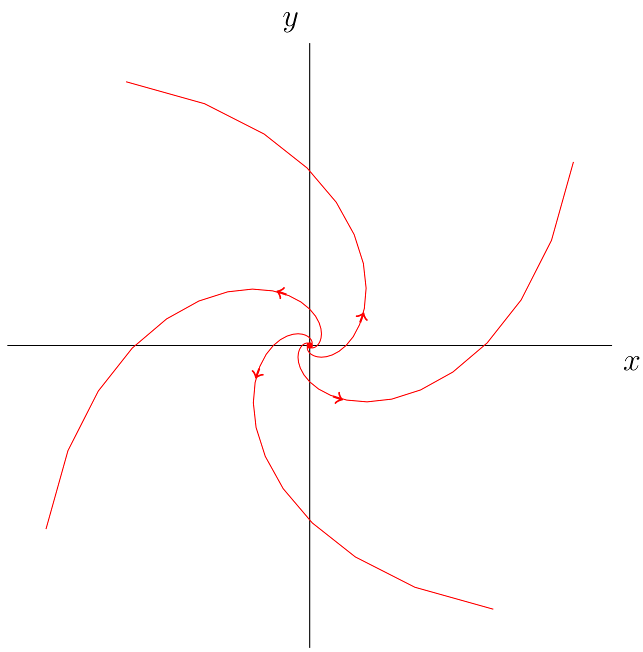

Figure 3: Sketch of phase portrait of system (13) with $\alpha > 0$ and $\beta = 1$