Solutions to Assignment #21

1. Let $f(x,y)=x^2+y^2$ for all $(x,y)\in\mathbb{R}^2$. Compute the directional derivative f $(2,1)$ in the direction of the line $y=x$ towards the first quadrant.

   Suggestion: Find a unit vector $\hat{u}$ in the direction of the line $y=x$ towards the first quadrant.

   ***Solution**:* We compute the directional derivative of f at $(2,1)$ in the direction of the vector
   $$\hat{u}=\frac{\sqrt{2}}{2}\ \hat{i}+\frac{\sqrt{2}}{2}\ \hat{j},$$
   $$D_{\hat{u}}f(2,1)=\nabla f(2,1)\cdot\hat{u},$$
   where
   $$\nabla f(x,y)=2x\ \hat{i}+2y\ \hat{j},\qquad \text{for all } (x,y)\in\mathbb{R}^2;$$
   so that,
   $$\nabla f(2,1)=4\ \hat{i}+2\ \hat{j}.$$
   Consequently,
   $$D_{\hat{u}}f(2,1)=\frac{\sqrt{2}}{2}(4)+\frac{\sqrt{2}}{2}(2)=3\sqrt{2}.$$
   □

2. The directional derivative of a function, f, of two variables, x and y, at $(2,1)$ in the direction towards the point $(1,3)$ is $-2/\sqrt{5}$, and the directional derivative at $(2,1)$ in the direction of towards the point $(5,5)$ is 1. Compute the first–order partial derivatives of f at $(2,1)$.

   ***Solution**:* We use the formula
   $$D_{\hat{u}}f(x_o,y_o)=\frac{\partial f}{\partial x}(x_o,y_o)\cdot u_1+\frac{\partial f}{\partial y}(x_o,y_o)\cdot u_2,$$
   where $(x_o,y_o)=(2,1)$ and u_1 and u_2 are the components of the unit vector $\hat{u}$; so that,
   $$D_{\hat{u}}f(2,1)=f_x(2,1)\ u_1+f_y(2,1)\ u_2. \tag{1}$$
   Let $u_1\ \hat{i}+u_2\ \hat{j}$ denote the unit vector in the direction from $(2,1)$ to $(1,3)$; then,
   $$u_1=-\frac{1}{\sqrt{5}}\quad\text{and}\quad u_2=\frac{2}{\sqrt{5}};$$

so that, using (1),

$$D_{\hat{u}}f(2,1) = -\frac{1}{\sqrt{5}}f_x(2,1) + \frac{2}{\sqrt{5}}f_y(2,1). \tag{2}$$

Similarly, if $v_1\ \hat{i} + v_2\ \hat{j}$ denotes the unit vector in the direction from $(2,1)$ to $(5,5)$, then

$$v_1 = \frac{3}{5} \quad \text{and} \quad v_2 = \frac{4}{5};$$

so that, using (1),

$$D_{\hat{v}}f(2,1) = \frac{3}{5}f_x(2,1) + \frac{4}{5}f_y(2,1). \tag{3}$$

We are given that

$$D_{\hat{u}}f(2,1) = -\frac{2}{\sqrt{5}} \tag{4}$$

and

$$D_{\hat{v}}f(2,1) = 1. \tag{5}$$

Thus, comparing (2) and (4),

$$-\frac{1}{\sqrt{5}}f_x(2,1) + \frac{2}{\sqrt{5}}f_y(2,1) = -\frac{2}{\sqrt{5}},$$

or

$$f_x(2,1) - 2f_y(2,1) = 2. \tag{6}$$

Similarly, comparing (3) and (5), we see that

$$\frac{3}{5}f_x(2,1) + \frac{4}{5}f_y(2,1) = 1,$$

or

$$3f_x(2,1) + 4f_y(2,1) = 5. \tag{7}$$

Thus, solving the equations in (6) and (7) we get that

$$f_x(2,1) = \frac{9}{5} \quad \text{and} \quad f_y(2,1) = -\frac{1}{10}.$$

□

3. A bug is moving on a two–dimensional plate, D, with temperature $u(x,y)$ for all $(x,y)\in D$. Assume that at $(x_o,y_o)\in D$,

$$\frac{\partial u}{\partial x}(x_o,y_o) = -2 \quad \text{and} \quad \frac{\partial u}{\partial y}(x_o,y_o) = 1.$$

Suppose the velocity of the bug at when it is at (x_o,y_o) is given by the vector $v = 4\hat{\imath} + 7\hat{\jmath}$. Compute the rate of change of temperature along the path of the bug at the point (x_o,y_o).

***Solution**:* Using the Chain–Rule we get

$$\frac{d}{dt}[u(x(t),y(t))] = \frac{\partial u}{\partial x}(x(t),y(t))\ \frac{dx}{dt} + \frac{\partial u}{\partial y}(x(t),y(t))\ \frac{dy}{dt},$$

for all $t\in\mathbb{R}$.

At a time t_o at which the bug is at (x_o,y_o), we get

$$\frac{d}{dt}[u(x(t),y(t))]\Big|_{t=t_o} = \frac{\partial u}{\partial x}(x_o,y_o)\ \frac{dx}{dt} + \frac{\partial u}{\partial y}(x_o,y_o)\ \frac{dy}{dt},$$

where

$$\frac{dx}{dt} = 4 \quad \text{and} \quad \frac{dy}{dt} = 7,$$

and

$$\frac{\partial u}{\partial x}(x_o,y_o) = -2 \quad \text{and} \quad \frac{\partial u}{\partial y}(x_o,y_o) = 1.$$

Thus, the rate of change of temperature along the path of the bug at the point (x_o,y_o) is

$$\begin{aligned}\frac{du}{dt}\Big|_{t=t_o} &= (-2)(4) + (1)(7) \\ &= -1.\end{aligned}$$

□

4. Let $\hat{u}$ denote a unit vector and put $\sigma(t) = x_o\hat{\imath} + y_o\hat{\jmath} + t\hat{u}$ for all $t\in\mathbb{R}$. Let $f\colon D\to\mathbb{R}$ be a real–valued function defined on some domain, D, in the xy–plane that contains the point (x_o,y_o).

   (a) Apply the Chain Rule to compute $\dfrac{d}{dt}[f(\sigma(t))]$ at $t=0$. Explain why this yields the directional derivative of f at (x_o,y_o) in the direction of $\hat{u}$.

***Solution**:* Let $\hat{u}$ denote a unit vector in $\mathbb{R}^2$ and define $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ by

$$\sigma(t) = (x_o, y_o) + t\hat{u}, \quad \text{for all } t \in \mathbb{R}. \tag{8}$$

Then, $\sigma(0) = (x_o, y_o)$ and, for $|t|$ sufficiently small, $\sigma(t) \in D$, and we can apply the Chain-Rule to conclude that $f \circ \sigma$ is differentiable and

$$\frac{d}{dt}[f(\sigma(t))] = \nabla f(\sigma(t)) \cdot \sigma'(t), \quad \text{for } |t| \text{ sufficeintly small,}$$

or, in view of (8),

$$\frac{d}{dt}[f((x_o, y_o) + t\hat{u})] = \nabla f((x_o, y_o) + t\hat{u}) \cdot \hat{u}, \tag{9}$$

for $|t|$ very close to 0.
Setting $t = 0$ in (9) we obtain

$$\frac{d}{dt}[f((x_o, y_o) + t\hat{u})]\Big|_{t=0} = \nabla f(x_o, y_o) \cdot \hat{u}. \tag{10}$$

The expression on the left–hand side of (10) gives the rate of change of the value of the function f at (x_o, y_o) along a line through (x_o, y_o) in the direction of $\hat{u}$ (this is the straight line parametrized by the path σ given in (8). It is therefore the **directional derivative** of f at (x_o, y_o) in the direction of the unit vector $\hat{u}$, which is also denoted by $D_{\hat{u}}f(x_o, y_o)$. Thus, (10) can be rewritten as

$$D_{\hat{u}}f(x_o, y_o) = \nabla f(x_o, y_o) \cdot \hat{u}.$$

□

(b) Deduce that

$$D_{\hat{u}}f(x, y) = \|\nabla f(x, y)\| \cos\theta, \quad \text{for all } (x, y) \in D, \tag{11}$$

where θ is the angle that $\nabla f(x, y)$ makes with the unit vector $\widehat{u}$.
Conclude from (11) that the rate of change of f at (x, y) is the largest in the direction of the gradient of f at (x, y).
***Solution**:* Since, according to (11), $D_{\hat{u}}f(x, y)$ is the dot product of the vector $\nabla f(x_o, y_o)$ and the vector $\hat{u}$, it follows that

$$D_{\hat{u}}f(x, y) = \|\nabla f(x, y)\|\|\hat{u}\| \cos\theta, \quad \text{for all } (x, y) \in D,$$

where θ is the angle between $\nabla f(x, y)$ and the unit vector $\hat{u}$. Thus, since $\|\hat{u}\| = 1$, (11) follows.
Observe that $\cos\theta$ is the largest possible when $\cos\theta = 1$. This happens when $\theta = 0$, or when $\hat{u}$ is in the same direction as $\nabla f(x, y)$. □

5. Let $f(x,y) = 3xy + y^2$ for all $(x,y) \in \mathbb{R}^2$.

   (a) Give the direction of maximum rate of change of f at $(2,3)$.

   ***Solution**:* According to the result in Problem 4, the direction of maximum rate of change of f at $(2,3)$ is that of the gradient of f at $(2,3)$. We therefore compute $\nabla f(2,3)$.

   The gradient of f at any (x,y) in $\mathbb{R}^2$ is given by

   $$\nabla f(x,y) = 3y\ \hat{\imath} + (3x+2y)\ \hat{\jmath}, \qquad \text{for all } (x,y) \in \mathbb{R}^2.$$

   Hence, The direction of maximum rate of change of f at $(2,3)$ is that of the vector

   $$\nabla f(2,3) = 9\ \hat{\imath} + 12\ \hat{\jmath}.$$

   □

   (b) Give the direction in which f is decreasing the fastest at $(2,3)$.

   ***Solution**:* According to (11, the rate of change of f at (x,y) in the direction of $\hat{u}$ is negative and the largest in magnitude then $\cos\theta = -1$. This occurs when θ is π radians, or $180°$. Thus, the direction of maximum decrease of f at $(2,3)$ is that opposite the gradient of f at $(2,3)$, or $-\nabla f(2,3)$. Hence, the direction in which f is decreasing the fastest at $(2,3)$ that of

   $$-\nabla f(2,3) = -9\ \hat{\imath} - 12\ \hat{\jmath}.$$

   □

   (c) Give the direction in which the rate of change of f is at $(2,3)$ is zero.

   ***Solution**:* In view of (11, we see that $D_{\hat{u}}f(2,3) = 0$ when $\cos\theta = 0$. This occurs when $\hat{u}$ is perpendicular to $\nabla f(2,3)$. Thus, the direction in which the rate of change of f is at $(2,3)$ is zero. is that of

   $$12\ \hat{\imath} - 9\ \hat{\jmath}$$

   or

   $$-12\ \hat{\imath} + 9\ \hat{\jmath}.$$

   □