Solutions to Assignment #5

1. Let $\widehat{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\widehat{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

   (a) Compute norms $\|\widehat{i}\|$ and $\widehat{j}\|$.
   Solution: Compute $\|\widehat{i}\| = \sqrt{1^2 + 0^2} = 1$.
   Similarly, $\|\widehat{j}\| = \sqrt{0^2 + 1^2} = 1$. □

   (b) Explain why $\widehat{i}$ and $\widehat{j}$ are perpendicular.
   Solution: The vector $\widehat{i}$ in standard position lies along the x–axis, while $\widehat{j}$ lies along the y–axis. Hence, $\widehat{i}$ and $\widehat{j}$ are perpendicular. □

   (c) Show that any vector v in $\mathbb{R}^2$ can be written as

   $$v = c_1\,\widehat{i} + c_2\,\widehat{j},$$

   for some real numbers c_1 and c_2.
   Solution: Let $v = \begin{pmatrix} a \\ b \end{pmatrix}$ be any vector in $\mathbb{R}^2$. Using the definitions of vector addition and scalar multiplication we can write

   $$v = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1}$$

   Thus, since

   $$\hat{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \hat{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2}$$

   we see from (1) that the vector v can be written as

   $$v = a\hat{i} + b\hat{j}. \tag{3}$$

   Setting $c_1 = a$ and $c_2 = b$ in (3), we get what we were asked to show. □

2. Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

   (a) Compute $v + w$ and sketch it in standard position.
   Solution: Compute

   $$v + w = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

   Figure 1 shows a sketch of $v + w$ in standard position. □

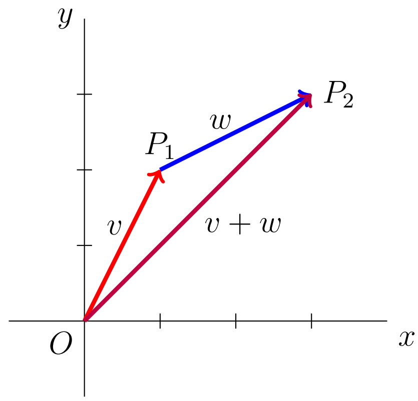

Figure 1: v, w and $v+w$

(b) Sketch v in standard position and sketch w with its starting point at the tip of v.

***Solution**:* See Figure 1. $\square$

(c) Verify that

$$\|v+w\| \leqslant \|v\| + \|w\|, \tag{4}$$

and explain why (4) is called the triangle inequality,

***Solution**:* Compute $\|v\| = \sqrt{1^2+2^2} = \sqrt{5}$, $\|w\| = \sqrt{2^2+1^2} = \sqrt{5}$, and

$$\|v+w\| = \sqrt{3^2+3^2} = 3\sqrt{2}.$$

Note that $\|v\| + \|w\| = 2\sqrt{5} \approx 4.47$ is bigger than $\|v+w\| = 3\sqrt{2} \approx 4.24$. Thus, the inequality in (4) is satisfied in this case.

Referring to the sketch in Figure 1, notice that the vectors v, w and $v+w$ are the sides of the triangle with vertices O, P_1 and P_2, where P_1 is the tip of v (drawn in standard position) and P_2 is the tip of $v+w$ (drawn in standard position). For this triangle to exist, the sum of the lengths of two of the sides must be grater than the length of the third side. $\square$

(d) Given an example of vectors v and w in $\mathbb{R}^2$ for which equality in (4) holds true.

***Solution**:* Let $v = \widehat{i}$ and $w = 2\widehat{i}$. Then, $v+w = 3\widehat{i}$. Thus, $\|v\| = \|\widehat{i}\| = 1$, $\|w\| = 2\|\widehat{i}\| = 2$, and $\|v+w\| = 3\|\widehat{i}\| = 3$. Hence,

$$\|v\| + \|w\| = 1 + 2 = 3 = \|v+w\|,$$

which shows that equality in (4) holds for this example. $\square$

3. Let v and w be as in Problem 2 and $\widehat{i}$ be as in Problem 4. Find real numbers c_1 and c_2 such that
$$c_1 v + c_2 w = \widehat{i}. \tag{5}$$

***Solution**:* Rewrite the equation in (5) in terms of the vectors given in Problem 2 and Problem 4 to get
$$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
from which we get
$$\begin{pmatrix} c_1 \\ 2c_1 \end{pmatrix} + \begin{pmatrix} 2c_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
or
$$\begin{pmatrix} c_1 + 2c_2 \\ 2c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{6}$$
Equating corresponding components of both sides of the equality in (6) yields the system of equations
$$\begin{cases} c_1 + 2c_2 &= 1; \\ 2c_1 + c_2 &= 0. \end{cases} \tag{7}$$

We can solve the system in (7) simultaneously by first solving for c_2 in the second equation in (7),
$$c_2 = -2c_1, \tag{8}$$
and then substituting into the first equation in (7) to get
$$c_1 + 2(-2c_1) = 1,$$
or
$$-3c_1 = 1;$$
so that,
$$c_1 = -\frac{1}{3} \tag{9}$$
Finally, substitute (9) into (8) to get
$$c_2 = \frac{2}{3}. \tag{10}$$

Hence, (9) and (10) give the values of c_1 and c_2, respectively, that will make the statement in (5) true. □

4. Let $\widehat{i}$ and $\widehat{j}$ be as in Problem 1.

   (a) Compute $\widehat{i} - \widehat{j}$ and $\|\widehat{i} - \widehat{j}\|$.
   Solution: Compute

$$\widehat{i} - \widehat{j} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

   Thus,

$$\|\widehat{i} - \widehat{j}\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}.$$

□

   (b) Sketch $\widehat{i}$ and $\widehat{j}$ in standard position and $\widehat{i} - \widehat{j}$ with its starting point at the tip of $\widehat{j}$.
   Solution: See the sketch in Figure 2. □

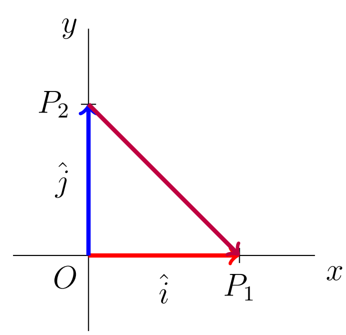

Figure 2: Sketch of $\hat{i}$, j and $\hat{i} - \hat{j}$

   (c) Verify that $\|\widehat{i} - \widehat{j}\|^2 = \|\widehat{i}\|^2 + \|\widehat{j}\|^2$. Give a geometric interpretation of this result.
   Solution: From part (a) we have that $\|\widehat{i} - \widehat{j}\|^2 = 2$. Since, $\|\hat{i}\| = 1$ and $\|\hat{j}\| = 1$, it follows that

$$\|\widehat{i} - \widehat{j}\|^2 = \|\widehat{i}\|^2 + \|\widehat{j}\|^2. \tag{11}$$

   The expression in (11) is a statement of the Pythagorean Theorem for the right triangle with vertices O, P_1 and P_2 pictured in Figure 2.

□

5. Let u be a vector in $\mathbb{R}^2$ or norm 1 and let v be any vector in $\mathbb{R}^2$.

(a) Give the vector–parametric equation of the line through origin in the direction of u.

Solution: The line through origin in the direction of u is parametrized by the vector equation

$$\sigma(t) = tu, \quad \text{for all } t \in \mathbb{R}.$$

□

(b) Let

$$f(t) = \|v - tu\|^2, \quad \text{for all } t \in \mathbb{R}. \tag{12}$$

Explain why this function gives the square of the distance from the point at v to a point on the line through the origin in the direction of u.

Solution: Figure 3 shows a sketch of the line through the origin in the direction of u. The line is labeled L in the sketch.

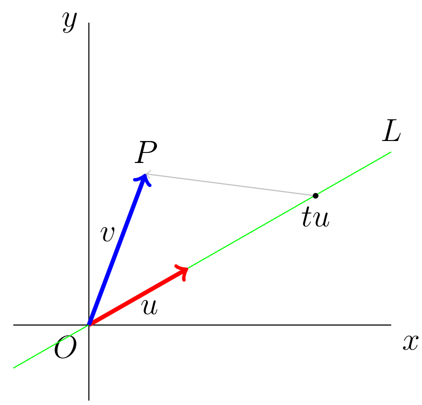

Figure 3: Sketch of line through the origin in the direction of u

Figure 3 also shows the vector v in standard position with its tip labeled P.

The distance from a typical point, tu, on L to the point P is the norm of the vector from tu to P, namely $\|v - tu\|$. Hence, the function f defined in (12) gives the square of the distance from the point at v to a point on the line through the origin in the direction of u. □

(c) Give the value of t at which $f(t)$ is minimized in terms of the components of u and v.

Solution: We need to find the value of t for which $f(t)$ given in (12) is the smallest possible. To do this, use the properties of the dot product and the Euclidean norm to compute

$$\begin{aligned} f(t) &= (v - tu) \cdot (v - tu) \\ &= v \cdot v - tv \cdot u - tu \cdot v + t^2 u \cdot u \\ &= \|v\|^2 - 2tv \cdot u + t^2\|u\|^2; \end{aligned}$$

so that, since u is a unit vector,

$$f(t) = \|v\|^2 - 2tv \cdot u + t^2, \quad \text{for all } t \in \mathbb{R}. \tag{13}$$

It follows from (13) that f is differentiable with derivatives

$$f'(t) = -2v \cdot u + 2t, \quad \text{for all } t \in \mathbb{R},$$

and

$$f''(t) = 2, \quad \text{for all } t \in \mathbb{R}.$$

Consequently, $f(t)$ is minimized when $f'(t) = 0$, or when

$$t = v \cdot u. \tag{14}$$

If $v = a\hat{\imath} + b\hat{\jmath}$ and $u = u_1\hat{\imath} + u_2\hat{\jmath}$, where a and b are the components of v, and u_1 and u_2 are the components of u; so that $u_1^2 + u_2^2 = 1$, then the expression for t in (14) reads

$$t = au_1 + bu_2.$$

□