Solutions to Assignment #6

1. Let $v = a\widehat{i} + b\widehat{j}$ be a vector in $\mathbb{R}^2$ such that $\|v\| \neq 0$.

(a) Give a vector $w \in \mathbb{R}^2$ that is orthogonal to v.
***Solution**:* Set $w = b\widehat{i} - a\widehat{j}$. Then,

$$v \cdot w = ab - ab = 0.$$

so that w is orthogonal to v. $\square$

(b) Give unit vectors $\widehat{v}$ and $\widehat{w}$ that are orthogonal to each other and such that $\widehat{v}$ is parallel to v and $\widehat{w}$ is parallel to w.
***Solution**:*

$$\widehat{v} = \frac{1}{\|v\|} v,$$

where $\|v\| = \sqrt{a^2 + b^2}$., and

$$\widehat{w} = \frac{1}{\|w\|} w,$$

where $\|w\| = \sqrt{b^2 + (-a)^2} = \sqrt{a^2 + b^2}$.
Hence,

$$\widehat{v} = \frac{a}{\sqrt{a^2 + b^2}} \widehat{i} + \frac{b}{\sqrt{a^2 + b^2}} \widehat{j}$$

and

$$\widehat{w} = \frac{b}{\sqrt{a^2 + b^2}} \widehat{i} - \frac{a}{\sqrt{a^2 + b^2}} \widehat{j}.$$

$\square$

(c) Let $\widehat{v}$ and $\widehat{w}$ be as in part (b). Put $u = c_1\widehat{v} + c_2\widehat{w}$, for some real numbers c_1 and c_2. Verify that

$$\|u\|^2 = c_1^2 + c_2^2. \tag{1}$$

Give and interpretation of this result.
***Solution**:* Let $u = c_1\widehat{v} + c_2\widehat{w}$ and compute

$$\begin{aligned} \|u\|^2 &= (c_1\widehat{v} + c_2\widehat{w}) \cdot (c_1\widehat{v} + c_2\widehat{w}) \\ &= c_1^2\widehat{v} \cdot \widehat{v} + c_1c_2\widehat{v} \cdot \widehat{w}) + c_2c_1\widehat{w} \cdot \widehat{v} + c_2^2\widehat{w} \cdot \widehat{w}; \end{aligned}$$

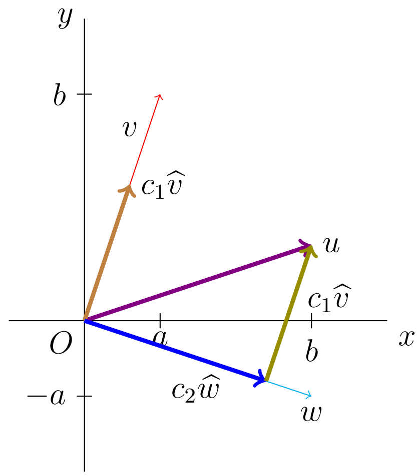

Figure 1: Vectors v and w

so that,

$$\|u\|^2 = c_1^2\|\widehat{v}\|^2 + 2c_1c_2\widehat{v}\cdot\widehat{w} + c_2^2\|\widehat{w}\|^2. \tag{2}$$

Now, since $\|\widehat{v}\| = \|\widehat{w}\| = 1$, and $\widehat{v}\cdot\widehat{w} = 0$, (1) follows from (2).

An interpretation of (1) can be seen in Figure 1. Consider the triangle with vertices at the origin, O, the tip of $c_2\widehat{w}$, and the tip of u shown in the Figure. Note that, by the parallelogram rule of vector addition, this triangle is a right triangle, since v and w are orthogonal. The hypotenuse of this triangle is u, of length $\|u\|$, and the legs of the triangle are $c_1\widehat{v}$, of length $|c_1|$, and $c_2\widehat{w}$, of length $|c_2|$. □

2. Let v and w denote vectors in $\mathbb{R}^2$.

   (a) Use the fact that $|\cos\theta| \leqslant 1$ for all $\theta \in \mathbb{R}$ to show that

   $$|v\cdot w| \leqslant \|v\|\|w\|. \tag{3}$$

   The statement in (3) is called the Cauchy–Schwarz inequality.

   ***Solution**:* Start with

   $$v\cdot w = \|v\|\|w\|\cos\theta, \tag{4}$$

   where θ is the angle between v and w.

   Take absolute value on both sides of (4) to get

   $$|v\cdot w| = \|v\|\|w\||\cos\theta|. \tag{5}$$

Then, since $|\cos\theta| \leqslant 1$, we get from (5) that

$$|v \cdot w| \leqslant \|v\|\|w\|,$$

which is the inequality in (3). □

(b) Determine conditions on the vectors v and w under which equality occurs in (3). Explain the reasoning leading to your answer.

Solution: Equality in (3) occurs when

$$|v \cdot w| = \|v\|\|w\|. \tag{6}$$

Comparing (6) and (5), we see that equality in (3) occurs when

$$|\cos\theta| = 1;$$

Thus, equality in (3) occurs when $\theta = 0$ or $\theta = \pi$. Hence, equality in (3) occurs when v and w lie on the same line. □

3. Use the Cauchy–Schwarz inequality in (3) to derive the **triangle inequality**:

$$\|v + w\| \leqslant \|v\| + \|w\|. \tag{7}$$

Suggestion: Compute $\|v + w\|^2 = (v + w) \cdot (v + w)$ using the properties of the dot product. Then, apply the Cauchy–Schwarz inequality.

Solution: Compute

$$\begin{aligned} \|v + w\|^2 &= (v + w) \cdot (v + w) \\ &= v \cdot v + v \cdot w + w \cdot v + w \cdot w; \end{aligned}$$

so that,

$$\|v + w\|^2 = \|v\|^2 + 2v \cdot w + \|w\|^2. \tag{8}$$

Now, since $v \cdot w \leqslant |v \cdot w|$, we obtain from (8) the inequality

$$\|v + w\|^2 \leqslant \|v\|^2 + 2|v \cdot w| + \|w\|^2. \tag{9}$$

Then, applying the Cauchy–Schwarz inequality to the middle term of the right–hand side of (9),

$$\|v + w\|^2 \leqslant \|v\|^2 + 2\|v\|\|w\| + \|w\|^2. \tag{10}$$

The right–hand side of (10) can be factored to yield the inequality

$$\|v + w\|^2 \leqslant (\|v\| + \|w\|)^2. \tag{11}$$

Finally, taking square roots on both sided of the inequality in (11) yields the triangle inequality in (7). □

4. Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

   (a) Explain why v and w are orthogonal.
   ***Solution**:* Compute the dot product

   $$v \cdot w = (1)(2) + (2)(-1) = 0.$$

   Thus, v and w are orthogonal. □

   (b) Give unit vectors $\widehat{v}$ and $\widehat{w}$ that are orthogonal to each other and such that $\widehat{v}$ is parallel to v and $\widehat{w}$ is parallel to w.
   ***Solution**:* Compute

   $$\widehat{v} = \frac{1}{\|v\|} v,$$

   where $\|v\| = \sqrt{1^2 + 2^2} = \sqrt{5}$; so that,

   $$\widehat{v} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}. \tag{12}$$

   Similarly,

   $$\widehat{w} = \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}. \tag{13}$$

   □

   (c) Given any vector $u = a\widehat{i} + b\widehat{j}$, find c_1 and c_2, in terms of a and b, such that

   $$u = c_1 \widehat{v} + c_2 \widehat{w}.$$

   c_1 is called the component of u along the direction of v and c_2 is the component of u along the direction of w.
   ***Solution**:* Start with the equation

   $$c_1 \widehat{v} + c_2 \widehat{w} = u. \tag{14}$$

Take the dot product with $\widehat{v}$ on both sides of (14) to get

$$(c_1\widehat{v} + c_2\widehat{w}) \cdot \widehat{v} = u \cdot \widehat{v};$$

so that, using the distributive property,

$$c_1\widehat{v} \cdot \widehat{v} + c_2\widehat{w} \cdot \widehat{v} = u \cdot \widehat{v}.$$

Then, since $\widehat{v} \cdot \widehat{v} = \|\widehat{v}\|^2 = 1$ and $\widehat{w} \cdot \widehat{v} = 0$,

$$c_1 = u \cdot \widehat{v}. \tag{15}$$

Similarly,

$$c_2 = u \cdot \widehat{w}. \tag{16}$$

To find c_1 and c_2 in (15) and (16), respectively, use the values of $\widehat{v})$ and $\widehat{w})$ in (12) and (13), respectively, along with the fact that $u = a\widehat{i} + b\widehat{j}$, to get

$$c_1 = \frac{a}{\sqrt{5}} + \frac{2b}{\sqrt{5}}$$

and

$$c_2 = \frac{2a}{\sqrt{5}} - \frac{b}{\sqrt{5}}.$$

□

5. Let J denote an open interval of real numbers, and let $\sigma \colon J \to \mathbb{R}^2$ and $\gamma \colon J \to \mathbb{R}^2$ be differentiable paths given by

$$\sigma(t) = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} \quad \text{and} \quad \gamma(t) = \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}, \quad \text{for } t \in J. \tag{17}$$

   (a) Define $f(t) = \sigma(t) \cdot \gamma(t)$, for $t \in J$. Use the definition of the dot product and the product rule to show that f is differentiable and give a formula for computing $f'(t)$.

   ***Solution**:* Use the definitions of σ and γ in (17) and the definition of the dot product to compute

$$f(t) = x_1(t)x_2(t) + y_1(t)y_2(t), \qquad \text{for } t \in J. \tag{18}$$

   Note that, according to (18), f is a sum of products of differentiable functions. Hence, by the product rule, f is differentiable and

$$f'(t) = x_1(t)x_2'(t) + x_1'(t)x_2(t) + y_1(t)y_2'(t) + y_1'(t)y_2(t), \qquad \text{for } t \in J,$$

or

$$f'(t) = x_1(t)x_2'(t) + y_1(t)y_2'(t) + x_1'(t)x_2(t) + y_1'(t)y_2(t), \quad \text{for } t \in J;$$

so that, using the definition of the dot product,

$$f'(t) = \sigma(t) \cdot \gamma'(t) + \sigma'(t) \cdot \gamma(t), \qquad \text{for } t \in J; \tag{19}$$

□

(b) Suppose that $\|\sigma(t)\| = C$, for all $t \in J$, and some constant C. Show that $\sigma'(t)$ is orthogonal to $\sigma(t)$ for all $t \in J$.

Suggestion: Write $\|\sigma(t)\|^2 = C^2$ in terms of the dot product to get

$$\sigma(t) \cdot \sigma(t) = C^2, \qquad \text{for all } t \in J. \tag{20}$$

Take the derivative with respect to t on both sides of the equation in (20) and use the result derived in part (a).

***Solution**:* Differentiate with respect to t on both sides of (20) to get

$$\frac{d}{dt}[\sigma(t) \cdot \sigma(t)] = 0, \qquad \text{for all } t \in J, \tag{21}$$

since C^2 is constant. Then, applying the formula in (19) on the left–hand side of (21),

$$\sigma(t) \cdot \sigma'(t) + \sigma'(t) \cdot \sigma(t) = 0, \qquad \text{for all } t \in J,$$

or

$$2\sigma(t) \cdot \sigma'(t) = 0, \qquad \text{for all } t \in J,$$

or

$$\sigma(t) \cdot \sigma'(t) = 0, \qquad \text{for all } t \in J,$$

which shows that $\sigma'(t)$ is orthogonal to $\sigma(t)$ for all $t \in J$. □