Solutions to Assignment #9

1. In this problem, you will sketch the flow of the vector field

$$F(x,y) = y\hat{\imath} + x\hat{\jmath}, \quad \text{for all } (x,y) \in \mathbb{R}^2. \tag{1}$$

The flow of the vector field in (1) are the solution curves of the system of differential equations

$$\begin{cases} \dot{x} &= y; \\ \dot{y} &= x. \end{cases} \tag{2}$$

(a) Use the expression

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}, \quad \text{for } \dot{x} \neq 0, \tag{3}$$

and the differential equations in (2) to obtain a differential equation involving only the variables x and y.

***Solution**:* Substituting the equations in (2) into (3) yields

$$\frac{dy}{dx} = \frac{x}{y}. \tag{4}$$

□

(b) Use separation of variables to solve the differential equations derived in part (a).

***Solution**:* The differential equation in (4) can be solved by separating variables:

$$y\ dy = x\ dx. \tag{5}$$

Integrating on both sides of (5),

$$\int y\ dy = \int x\ dx,$$

yields

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + c_1, \tag{6}$$

for some constant of integration c_1.

Multiply on both sides of the equation by 2 and setting $C = -2c_1$, we obtain from (6) that

$$x^2 - y^2 = C. \tag{7}$$

□

(c) Sketch all possible solution curves obtained in part (b).

Solution: We consider three cases in (7):

(i) $C = 0$,

(ii) $C > 0$, and

(iii) $C < 0$.

In the case $C = 0$, (7) yields

$$x^2 - y^2 = 0,$$

or

$$(x + y)(x - y) = 0,$$

from which we get the two equations

$$y = x \quad \text{or} \quad y = -x. \tag{8}$$

Note that the lines in (8) correspond to five trajectories in the flow of the vector field in (1): The point where the two lines in (8) meet corresponds the equilibrium point $(0, 0)$ of the system in (2). This is shown as a dot in Figure 1. The other other four trajectories correspond to the portions of the lines in each of the four quadrants. These are shown in the sketch in Figure 1 with their respective directions indicated on them.

In the case in which $C > 0$, the graph of the equation in (7) consists of hyperbolas with x–intercepts $\pm\sqrt{C}$. Each of the branches of the the hyperbolas correspond to different trajectories of the system in (2). Four of those trajectories are sketched in Figure 1 with the directions indicated on them.

Finally, in the case $C < 0$, the graph of the equation in (7) consists of hyperbolas with y–intercepts $\pm\sqrt{-C}$. Each of the branches of the the hyperbolas correspond to different trajectories of the system in (2). Four of those trajectories are sketched in Figure 1 with the directions indicated on them. □

(d) Indicate the directions along the solution curves of the system in (2) in the sketch obtained in part (c).

Solution: To determine the direction of the trajectories sketched in Figure 1, look at the signs of $\dot{x}$ and $\dot{y}$ given by the differential equations in each of the four quadrants; these, are shown in Figure 1. For instance , in the first quadrant, $\dot{x} > 0$ and $\dot{y} > 0$; thus, x and y increase as t increases; consequently, the direction on the trajectories in the first quadrant is to

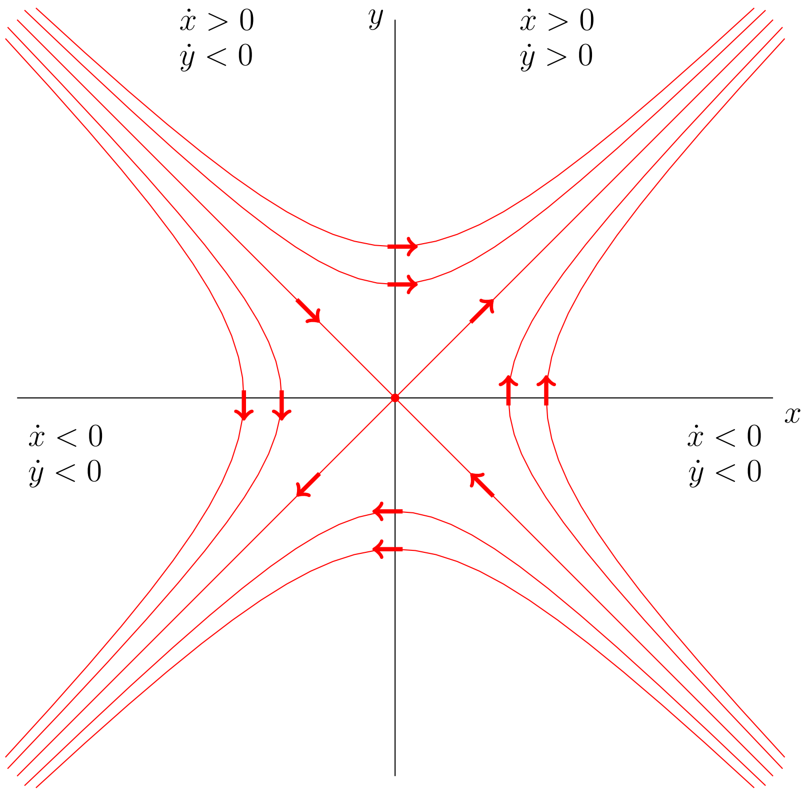

Figure 1: Sketch of Flow of Vector Field

the right and upwards. Thus the trajectory on the portion of the line $y = x$ in the first quadrant appears to be emanating from the origin and moving away from it. By the same token, in the second quadrant, $\dot{x} > 0$ and $\dot{y} < 0$; so that, x increases and y decreases at t increases; thus, the direction on the trajectories in the second quadrant is to the right and downwards. Thus the trajectory on the portion of the line $y = -x$ in the second quadrant appears to be tending towards the origin. □

2. **The Hyperbolic Functions**. The hyperbolic cosine function, denoted by cosh, is the function $\cosh\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\cosh(t) = \frac{e^t + e^{-t}}{2}, \qquad \text{for } t \in \mathbb{R}; \tag{9}$$

and the hyperbolic sine function, denoted $\sinh\colon \mathbb{R} \to \mathbb{R}$, is defined by

$$\sinh(t) = \frac{e^t - e^{-t}}{2}, \qquad \text{for } t \in \mathbb{R}. \tag{10}$$

Let $x(t) = \cosh(t)$ and $y(t) = \sinh(t)$, for all $t \in \mathbb{R}$, where cosh and sinh are defined in (9) and (10), respectively.

(a) Verify that $\dot{x} = y$ and $\dot{y} = x$.

Solution: Use (9) to compute

$$\begin{aligned} \dot{x} &= \frac{d}{dt}[\cosh(t)] \\ &= \frac{d}{dt}\left[\frac{e^t + e^{-t}}{2}\right] \\ &= \frac{e^t - e^{-t}}{2}; \end{aligned}$$

so that, in view of (10),

$$\dot{x} = \sinh(t) = y,$$

which was to be shown.

Similarly, using (10), we compute

$$\begin{aligned} \dot{y} &= \frac{d}{dt}[\sinh(t)] \\ &= \frac{d}{dt}\left[\frac{e^t - e^{-t}}{2}\right] \\ &= \frac{e^t + e^{-t}}{2}; \end{aligned}$$

so that, in view of (9),

$$\dot{y} = \cosh(t) = x,$$

which was to be shown. □

(b) Verify that $x^2 - y^2 = 1$.

Solution: Use (9) and (10) to compute

$$\begin{aligned} x^1 - y^2 &= (\cosh(t))^2 - (\sinh(t))^2 \\ &= \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2; \end{aligned}$$

so that,

$$x^1 - y^2 = \frac{(e^t + e^{-t})^2}{4} - \frac{(e^t - e^{-t})^2}{4},$$

or

$$\begin{aligned} x^1 - y^2 &= \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4} \\ &= \frac{e^{2t} + 2 + e^{-2t} - e^{2t} + 2 - e^{-2t}}{4} \\ &= \frac{4}{4}; \end{aligned}$$

so that, $x^2 - y^2 = 1$, which was to be shown. □

(c) Sketch the curve parametrized by

$$\sigma(t) = x(t)\hat{i} + y(t)\hat{j}, \quad \text{for all } t \in \mathbb{R}.$$

Indicate the direction given by the parametrization in the sketch.

***Solution**:* See the sketch in Figure 2. □

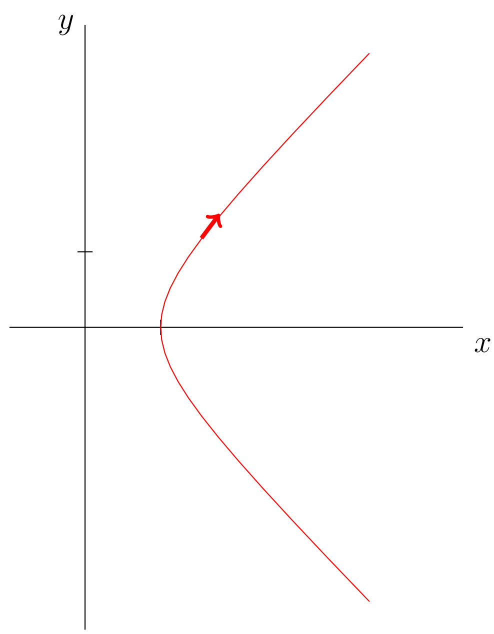

Figure 2: Sketch of $\sigma(t)$

(d) Give the equation of the tangent line to the curve at the point $(1, 0)$.

***Solution**:* The point $(1, 0)$ corresponds to $t = 0$. The direction of the tangent line to the path at $(1,0)$ is $\sigma'(0)$, where

$$\begin{aligned}\sigma'(t) &= \dot{x}(t)\hat{\imath} + \dot{y}(t)\hat{\jmath}\\ &= \sinh(t)\hat{\imath} + \cosh(t)\hat{\jmath};\end{aligned}$$

so that,

$$\sigma'(0) = \sinh(0)\hat{\imath} + \cosh(0)\hat{\jmath} = \hat{\jmath}.$$

Thus, the vector–parametric equation of the tangent line to the path σ at $(1,0)$ is

$$\ell(t) = \begin{pmatrix}1\\0\end{pmatrix} + t\begin{pmatrix}0\\1\end{pmatrix}, \quad \text{for } t \in \mathbb{R}.$$

or

$$\ell(t) = \begin{pmatrix}1\\t\end{pmatrix}, \quad \text{for } t \in \mathbb{R}.$$

This is equivalent to to the parametric equations

$$\begin{cases} x = 1; \\ y = t, \end{cases} \quad \text{for } t \in \mathbb{R},$$

or the vertical line $x = 1$. □

3. Consider a differentiable path $\sigma\colon J \to \mathbb{R}^2$ given by $\sigma(t) = \begin{pmatrix}x(t)\\y(t)\end{pmatrix}$, for $t \in J$, where J is an open interval

   Let $r(t)$ denote the norm of $\sigma(t)$ for all $t \in J$ and $\theta(t)$ denote the angle that $\sigma(t)$ makes with the positive x–axis.

   (a) Give formulas for computing $r(t)$ and $\theta(t)$, for $t \in J$, in terms of $x(t)$ and $y(t)$ for $t \in J$.

   ***Solution**:* Refer to the sketch in Figure 3.

   The formula for $r(t)$,

$$r(t) = \sqrt{(x(t))^2 + (y(t))^2}, \quad \text{for all } t \in J, \tag{11}$$

   can be obtained from the sketch in the figure by applying the Pythagorean Theorem, or by using the definition of the Euclidean norm.

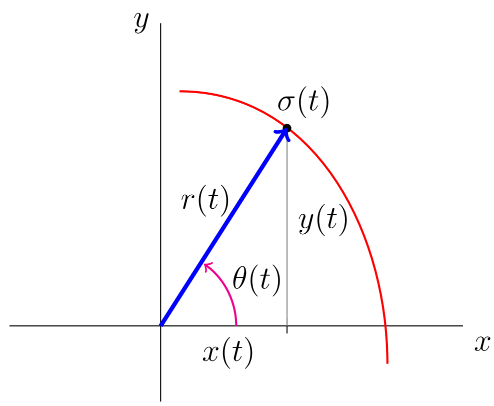

Figure 3: Sketch of path $\sigma(t)$

By the same token, using properties of right triangles and the definitions of trigonometric functions, we obtain that

$$\tan(\theta(t)) = \frac{y(t)}{x(t)}, \tag{12}$$

provided that $x(t) \neq 0$, from which we get

$$\theta(t) = \arctan\left(\frac{y(t)}{x(t)}\right), \quad \text{provided } x(t) \neq 0. \tag{13}$$

□

(b) Explain why the equations

$$\begin{cases} x(t) &= r(t)\cos(\theta(t)); \\ y(t) &= r(t)\sin(\theta(t)), \end{cases} \quad \text{for } t \in J, \tag{14}$$

are true.

***Solution**:* Refer to the sketch in Figure 3 and use the definitions of the trigonometric functions in right triangles to obtain the equations in (14). Indeed, the triangles with sides of lengths $x(t)$, $y(t)$ and $r(t)$ in the figure is a right with hypotenuse of length $r(t)$. The side adjacent to the angle $\theta(t)$ has length $x(t)$ and the opposite side has length $y(t)$, as shown in Figure 3. □

4. Let σ, r and θ be as defined in Problem 3.

   Assume that $\sigma(t)$ is not the zero vector for all $t \in J$. Use the formulas in derived in Problem 3 to explain why r and θ are differentiable functions of t, and verify that

$$\begin{cases} \dot{r} &= \dfrac{\dot{x}}{r} \cdot x + \dfrac{\dot{y}}{r} \cdot y, \\ \\ \dot{\theta} &= \dfrac{\dot{y}}{r^2} \cdot x - \dfrac{\dot{x}}{r^2} \cdot y. \end{cases} \qquad (15)$$

   Suggestion: Begin with the equations $r^2 = x^2 + y^2$ and $\tan\theta = \dfrac{y}{x}$; differentiate on both sides with respect to t; and apply the Chain Rule.

   Solution*:* Since we are assuming that $\sigma(t)$ is not the zero vector for any $t \in J$, it follows from (11) and the Chain Rule that the function r given in (11) is a differentiable function of t because it is a composition of differentiable functions.

   Similarly, in view of (13), we see that θ is a differentiabld function of t, by virtue of the Chain Rule.

   We get from (11) that

$$r^2 = x^2 + y^2, \qquad (16)$$

   and from (12), or (13), that

$$\tan\theta = \frac{y}{x}. \qquad (17)$$

   We would like to obtain expressions for the derivatives of r and t with respect to t in terms of $\dot{x}$ and $\dot{y}$.

   Taking the derivative with respect to t on both sides of the expression in (16) and using the Chain Rule, we obtain

$$2r\frac{dr}{dt} = 2x\dot{x} + 2y\dot{y},$$

   from which we get

$$\frac{dr}{dt} = \frac{1}{r}(x\dot{x} + y\dot{y}), \qquad \text{for } r > 0. \qquad (18)$$

   Similarly, taking the derivative with respect to t on both sides of (17) and applying the Chain Rule and the Quotient Rule,

$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{x\dot{y} - y\dot{x}}{x^2}, \qquad \text{for } x \neq 0;$$

so that, using the trigonometric identity

$$1 + \tan^2\theta = \sec^2\theta$$

and (17),

$$\left(1 + \frac{y^2}{x^2}\right)\frac{d\theta}{dt} = \frac{x\dot{y} - y\dot{x}}{x^2}, \qquad \text{for } x \neq 0. \tag{19}$$

Next, multiply both sides of the equation in (19) by x^2, for $x \neq 0$, to get

$$\left(x^2 + y^2\right)\frac{d\theta}{dt} = x\dot{y} - y\dot{x};$$

so that, in view of (16),

$$\frac{d\theta}{dt} = \frac{1}{r^2}(x\dot{y} - y\dot{x}), \qquad \text{for } r > 0. \tag{20}$$

Combine the equations in (18) and (20) to obtain the change of variables equations

$$\begin{cases} \dot{r} &= \dfrac{1}{r}(x\dot{x} + y\dot{y}); \\ \dot{\theta} &= \dfrac{1}{r^2}(x\dot{y} - y\dot{x}). \end{cases} \tag{21}$$

Note that the equations in (21) are the equations in (15), which we were asked to derive. □

5. In this problem we find the solutions of the system

$$\begin{cases} \dot{x} &= -\beta y; \\ \dot{y} &= \beta x, \end{cases} \tag{22}$$

where $\beta > 0$.

(a) Assume the equations in (22) are true, and use the equations in (15) to obtain a system of the form

$$\begin{cases} \dot{r} &= f(r, \theta); \\ \dot{\theta} &= g(r, \theta), \end{cases} \tag{23}$$

for some functions f and g that depend on r and θ.

***Solution**:* Substitute the expressions for $\dot{x}$ and $\dot{y}$ given by the right–hand sides of the equations in (22) into the right–hand side of the first equation in (21) to get

$$\dot{r} = \frac{1}{r}(x(-\beta y) + y(\beta x) = 0,$$

or

$$\dot{r} = 0. \tag{24}$$

Similarly, substituting the expressions for $\dot{x}$ and $\dot{y}$ given by the right–hand sides of the equations in (22) into the right–hand side of the second equation in (21) yields

$$\dot{\theta} = \frac{1}{r^2}(x(\beta x) - y(-\beta y)) = \frac{\beta}{r^2}(x^2 + y^2);$$

so that, in view of (16),

$$\dot{\theta} = \beta. \tag{25}$$

Putting together the equations in (24) and (25) yields the system

$$\begin{cases} \dot{r} &= 0; \\ \\ \dot{\theta} &= \beta. \end{cases} \tag{26}$$

Note that the system in (26) is in the form of the system in (23) with $f(r, \theta) = 0$ and $g(r, \theta) = \beta$. □

(b) Solve the system in (23).

***Solution**:* The system in (26) can be integrated to yield

$$\begin{cases} r(t) &= a; \\ \\ \theta(t) &= \beta t + \phi, \end{cases} \qquad \text{for } t \in \mathbb{R}, \tag{27}$$

where a and ϕ are constants of integration. □

(c) Use the solutions obtained in part (b) and the equations in (14) to obtain solutions of the system in (22).

***Solution**:* The expressions for r and θ in (27) can now be used, in conjunction with the equations in (14), to yield the solutions of the the system in (22):

$$\begin{cases} x(t) &= a\cos(\beta t + \phi); \\ \\ y(t) &= a\sin(\beta t + \phi), \end{cases} \qquad \text{for } t \in \mathbb{R}.$$

□